

# THE CAUCHY PROBLEM FOR THE HOMOGENEOUS MONGE-AMPÈRE EQUATION, II. LEGENDRE TRANSFORM

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**ABSTRACT.** We continue our study of the Cauchy problem for the homogeneous (real and complex) Monge-Ampère equation (HRMA/HCMA). In the prequel [RZ2] a quantum mechanical approach for solving the HCMA was developed, and was shown to coincide with the well-known Legendre transform approach in the case of the HRMA. In this article—that uses tools of convex analysis and can be read independently—we prove that the candidate solution produced by these methods ceases to solve the HRMA, even in a weak sense, as soon as it ceases to be differentiable.

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## 1. INTRODUCTION AND MAIN RESULTS

This article is the second in a series whose aim is to study existence, uniqueness and regularity of solutions of the Cauchy problem for the homogeneous (real and complex) Monge-Ampère equation (HRMA/HCMA). Our goal in the present article is to show that the well-known Legendre transform method for linearizing the HRMA fails to solve the equation as soon as the solution of the linear equation fails to be convex.

The Cauchy problem studied in this article corresponds to the initial value problem (IVP) for geodesics in the space of Kähler metrics. The IVP can be phrased as a Cauchy problem for the HCMA on the product  $S_T \times M$  of a strip,  $S_T := [0, T] \times \mathbb{R}$ , and a Kähler manifold  $M$ , and in the presence of an  $(S^1)^n$  symmetry the HCMA reduces to the HRMA on  $[0, T] \times \mathbb{R}^n$ .

In the first part [RZ2], we constructed a certain *quantum analytic continuation potential* on any projective Kähler manifold  $M$  and conjectured that it solved the IVP for as long as a solution exists. We evaluated it explicitly in the case of torus invariant metrics on toric and Abelian varieties, and showed that it equals the well-known Legendre transform potential in those cases. The main purpose of this sequel is to investigate the lifespan of this Legendre transform potential. An important part of the analysis is to determine rather precise regularity estimates for this function. Our main result (Theorem 1) shows that it ceases to solve the HRMA, even as a weak solution, as soon as it develops singularities, or equivalently, when the associated solution to the linear problem ceases to be convex. At the same time, we show that it does solve the equation on its dense regular locus. The proof of Theorem 1 uses only tools of convex analysis and is independent of the quantum techniques of [RZ2].

The Cauchy problem for HRMA studied in this article is given by

$$\left\{ \begin{array}{ll} \text{MA } \psi = 0, & \text{on } [0, T] \times \mathbb{R}^n, \\ \psi(0, \cdot) = \psi_0(\cdot), & \text{on } \mathbb{R}^n, \\ \frac{\partial \psi}{\partial s}(0, \cdot) = \dot{\psi}_0(\cdot), & \text{on } \mathbb{R}^n. \end{array} \right. \quad (1)$$

Here, MA denotes the real Monge-Ampère operator that associates a Borel measure to a convex function (see §§2.2) and equals  $\det \nabla^2 f \, dx^1 \wedge \cdots \wedge dx^{n+1}$  on  $C^2$  functions.

It is well-known that the HRMA is linearized by the Legendre transform. In geometric terms, the Legendre transform is an isometry between the space of open-orbit Kähler potentials and the space of symplectic potentials. Both spaces are flat, however the latter also has a trivial connection (see [RZ1], §3). Therefore, a geodesic  $\{\psi_s\}$  of open-orbit Kähler potentials, i.e., a solution of (1) with  $\psi_s = \psi(s, \cdot)$ , is transformed under the Legendre transform to a straight line of symplectic potentials

$$\psi_s^* = u_s = u_0 + s\dot{u}_0 \quad (2)$$

defined on the polytope  $P = \overline{\text{Im } \nabla \psi_s}$  corresponding to the toric Kähler class (see §§2.3 for more background, as well as [RZ1], §3, and [RZ2], §§4.2). There exists a

certain (typically) finite time  $T_{\text{span}}^{\text{cvx}}$ , that we call *the convex lifespan*, such that (1) restricted to  $[0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n$  may be linearized and hence solved via the Legendre transform. One has

$$T_{\text{span}}^{\text{cvx}}(\psi_0, \dot{\psi}_0) := \sup\{s : \psi_0^* - s\dot{\psi}_0 \circ (\nabla \psi_0)^{-1} \text{ is convex on } \text{Im } \nabla \psi_0\}. \quad (3)$$

The corresponding solution, that we call the *Legendre transform potential*, can be written explicitly as,

$$\psi(s, x) = \psi_s(x) := (u_0 + s\dot{u}_0)^*(x), \quad (s, x) \in [0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n. \quad (4)$$

The standard proofs that  $\psi$  solves the HRMA when  $u = u(s, \cdot)$  solves the linear equation  $\ddot{u} = 0$ , break down as soon as  $\psi$  is not twice differentiable. The underlying reason is that the Legendre transform is not a bijection on non-convex functions. Hence, as soon as  $u_s$  ceases to be convex on the polytope  $P$ , one may not deduce—at least not using the classical reasoning—that  $\psi$  solves the HRMA.

Nevertheless, one may consider the formula (4) as defining a convex function (still denoted by  $\psi$ ) on all of  $\mathbb{R}_+ \times \mathbb{R}^n$ . A natural question is whether  $\psi$  continues to solve the HRMA (1) for  $T > T_{\text{span}}^{\text{cvx}}$  in a weak sense.

The main result of this article is that  $\psi$  does not solve the HRMA (1) for any  $T > T_{\text{span}}^{\text{cvx}}$ . At the same time, we prove that  $\psi$  does solve the equation wherever it is differentiable, and in particular on a dense set whose complement has zero Lebesgue measure. To state our result, let

$$\Delta(\psi) := \{(s, x) : \psi \text{ is finite and differentiable at } (s, x)\} \subset \mathbb{R}_+ \times \mathbb{R}^n,$$

denote the regular locus of  $\psi$ . Similarly, we denote the regular locus of  $\psi_s$  by  $\Delta(\psi_s) \subset \mathbb{R}^n$ . Let

$$\Sigma_{\text{sing}} := \mathbb{R}_+ \times \mathbb{R}^n \setminus \Delta(\psi),$$

denote the singular locus of  $\psi$ , and set

$$\Sigma_{\text{sing}}(T) := \Sigma_{\text{sing}} \cap [0, T] \times \mathbb{R}^n.$$

Since  $\psi$  is everywhere finite the singular locus of  $\psi$  has Lebesgue measure zero, and the regular locus is dense in  $\mathbb{R}_+ \times \mathbb{R}^n$

**THEOREM 1.** *Let  $\psi$  be defined by (4) for all  $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ . Then*

(i)  *$\psi$  solves the HRMA (1) on the regular locus. Namely,*

$$\text{MA}\psi = 0 \text{ on } \Delta(\psi).$$

*In addition,  $[0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n \subset \Delta(\psi)$ .*

(ii) *Whenever  $T > T_{\text{span}}^{\text{cvx}}$ ,  $\psi$  fails to solve the HRMA (1). In particular, the Monge-Ampère measure of  $\psi$  charges the set  $\Sigma_{\text{sing}}(T)$  with positive mass and we have,*

$$\int_{[0, T] \times \mathbb{R}^n} \text{MA}\psi = \int_{\Sigma_{\text{sing}}(T)} \text{MA}\psi > 0.$$

Let us outline the main steps in the proof. In addition, a concrete overview of the proof is given in §3 for the case  $n = 1$ .

In order for a convex function to be a weak solution of the HRMA the image of its subdifferential must be a set of Lebesgue measure zero. Our goal is therefore

to obtain some description of the subdifferential mapping  $\partial\psi$ . First, we study the regularity of the restriction  $\psi_s$  of  $\psi$  to each time slice. Let

$$A_s := \{y \in P : u_s(y) \neq u_s^{**}(y)\} \subset P. \quad (5)$$

**PROPOSITION 1.** *For each  $s > 0$ , the function*

$$\psi_s(x) := (u_0 + s\dot{u}_0)^*(x), \quad x \in \mathbb{R}^n,$$

*is a continuous strictly convex function on  $\mathbb{R}^n$ . It is Lipschitz continuous but not everywhere differentiable. The singular locus of  $\psi_s$  is given by*

$$\nabla u_s^{**}(A_s \setminus \partial P).$$

Geometrically, Proposition 1 implies that  $\psi$  can be regarded as an infinite ray in the space of Lipschitz continuous open-orbit Kähler potentials. Its proof is completed at the end of §10, and is divided into several steps. We also remark that the strict convexity is not directly used for the proof of Theorem 1, however several of the ingredients in its derivation are.

First, in order to prove that  $\psi_s$  is strictly convex we study the regularity of its dual  $u_s^{**}$ . We prove interior  $C^1$  regularity for  $u_s^{**}$  (Lemma 4.1). Then we prove that  $u_s^{**}$  is moreover essentially smooth (Lemma 6.1), i.e., its gradient blows-up on  $\partial P$ . These results, together with a classical duality result, then imply the strict convexity of  $\psi_s$ . Here one uses the fact that  $\psi_s$  is everywhere finite, and so its subdifferential has domain equal to  $\mathbb{R}^n$ , and also  $\text{Im } \partial u_s^{**} = \mathbb{R}^n$ . These facts can be seen directly for our explicit  $\psi_s$ , but we also include a different proof (Lemma 7.2) using a homotopy argument that may have its own interest, and also shows as a by-product that  $u_s^{**}$  and  $\nabla u_s^{**}$  are continuous in  $s$ , a fact that is used later.

Second, to show that  $\psi_s$  is not differentiable we show that  $u_s^{**}$  is not strictly convex. To that end, for each  $s > T_{\text{span}}^{\text{cvx}}$ , we consider the set  $A_s \subset P$  defined in (5). We show that this set can be partitioned into convex sets

$$\overline{A_s} \setminus \partial P = \bigcup_{v \in A_s} Q(s, v)$$

along which  $\nabla u_s^{**}$  is constant (§9 and §13). Each such convex set  $Q(s, y)$  (see (76)) contains at least a line, and moreover if we let  $x = \nabla u_s^{**}(y)$  then  $\partial_x \psi(s, x) = Q(s, y)$  (Lemma 12.4), proving that  $\psi_s$  is not differentiable at  $x$ . We then obtain the exact description of the regular locus of  $\psi_s$  in terms of  $u_s$  and  $A_s$  (Lemma 10.3), and this concludes the proof of Proposition 1.

As just described, our results on the functions  $u_s^{**}$  and their gradient maps already give a precise description of the singularities of each  $\psi_s$ . Next, we prove a partial  $C^1$  regularity result for  $\psi$  (Lemma 10.4) that, as a corollary, gives a precise description of the singularities of  $\psi$ . Recall that  $\Delta(f)$  denotes the set on which  $f$  is finite and differentiable.

**PROPOSITION 2.** *Assume that  $x \in \Delta(\psi_s)$ . Then  $(s, x) \in \Delta(\psi)$ . I.e., the singular locus of  $\psi$  is the (indexed) union over  $s$  of the singular loci of the functions  $\psi_s$ . In terms of the regular loci,*

$$\Delta(\psi) = \bigcup_{s \geq T_{\text{span}}^{\text{cvx}}} \{s\} \times \Delta(\psi_s).$$

In other words, wherever  $\psi$  is differentiable in  $x$  it is also differentiable in  $s$ . The results described so far then imply an alternative description for the regular locus  $\Delta(\psi)$  in terms of the maps  $\partial u_s$ ,  $s \in \mathbb{R}_+$  (Proposition 10.1). This allows us to show that  $\psi$  solves the HRMA on the regular locus. In fact, the image of the (total) sub-differential of  $\psi$  evaluated on the regular locus is just the graph of  $-\dot{u}_0$  over  $P \setminus \partial P$ , and this, as a set in  $\mathbb{R}^{n+1}$ , has Lebesgue measure zero (Proposition 11.1).

To conclude the proof it thus remains to analyze the Monge-Ampère measure  $\text{MA}\psi$  on the singular locus. First, using some elementary facts regarding partial subdifferentials of convex functions of several variables we give a description of the  $x$  partial subdifferential of  $\psi$  and of the set of reachable  $x$  partial subgradients of  $\psi$  in terms of  $u_s$  and the sets  $Q(s, v)$  in the partition of  $A_s$  (Lemma 12.4). Then, by using the partial regularity of  $\psi$  we obtain some lower and upper bounds on  $\partial\psi(\{s\} \times \mathbb{R}^n)$  (Lemma 12.5).

**LEMMA 1.** *Let  $y \in A_s \setminus \partial P$ , and let  $x := \nabla u_s^{**}(y)$ . Then*

$$\text{co} \{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s, x)\} \subset \partial\psi(s, x) \subset \text{co} \{(-\dot{u}_0(v), v) : v \in \partial_x \psi(s, x)\}.$$

These bounds are obtained in terms of certain convex sets projecting onto the pieces  $Q(s, v)$  of the partition of  $A_s$ . Using these bounds, along with monotonicity and continuity of the family of the one-parameter family of sets  $A_s$  (Lemmas 8.1 and 8.2), we show that the subdifferential of  $\psi$  in fact “fills-in” a portion of the region lying between the graph of  $-\dot{u}_0$  and the graph of minus the convexification of  $\dot{u}_0$  (see Figures 3 and 4), hence the mass of  $\text{MA}\psi$  necessarily becomes positive for any  $T > T_{\text{span}}^{\text{cvx}}$  (Proposition 13.1 and Lemma 13.3), completing the proof.

It is worth pointing out that the proof also shows that the Monge-Ampère mass of  $\psi$  has an a priori upper bound depending only on the Cauchy data (Lemma 13.3). Let  $\text{epi } f$  denote the epigraph of  $f$  (see §§2.1).

**PROPOSITION 3.** *Let  $T > 0$ . One has,*

$$\int_{[0, T] \times \mathbb{R}^n} \text{MA } \psi \leq \text{Vol}(\text{epi}(-\dot{u}_0) \setminus \text{epi}(-(\dot{u}_0)^{**})),$$

where the right hand side denotes the volume in  $\mathbb{R}^{n+1}$  of the set of points lying below the graph of  $\dot{u}_0$  and above the graph of its convexification, over  $P$ .

The graph of  $\psi$  defines a hypersurface in  $\mathbb{R}^{n+2}$  that is flat over  $[0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n$ . Proposition 3 implies that the non-compact surface in  $\mathbb{R}^{n+2}$  defined by the graph of  $\psi$ , while not flat, has finite total Gauss curvature (see §§1.4 and §3).

Note that the right hand side is zero if and only if  $\dot{u}_0$  is convex. The inequality is also sharp when the right hand side is positive. This can be seen by considering,

for instance, explicit examples in  $n = 1$  (see §3), where equality is attained in the limit where  $T$  tends to infinity.

It is interesting to give a geometric description of the singular locus  $\Sigma_{\text{sing}}$  on which the Monge-Ampère mass is concentrated. By Propositions 1 and 2 we have an explicit description of the singular locus,

$$\Sigma_{\text{sing}} = \bigcup_{s \geq T_{\text{span}}^{\text{cvx}}} \{s\} \times \nabla u_s^{**}(A_s \setminus \partial P). \quad (6)$$

Now, by the continuity of  $\nabla u_s^{**}$  in  $s$  and the set-valued continuity of the sets  $A_s$ , it follows that  $\Sigma_{\text{sing}}$  is a countable union of  $C^0$  hypersurfaces in  $\mathbb{R}^{n+1}$ . Moreover, by general results of Alberti [A] in fact  $\Sigma_{\text{sing}}$  must be a countable union of locally Lipschitz continuous hypersurfaces in  $\mathbb{R}^{n+1}$ . Further, for reasonable Cauchy data, e.g., such that  $A_s$  has finitely many components for all  $s > 0$ , it follows from (6) that  $\Sigma_{\text{sing}}$  will also be composed of finitely many hypersurfaces. A visualization of the singular set and the corresponding ‘corner set’ of the graph of  $\psi$  is given in §3 (see Figures 2 and 3).

**1.1. Consequences for the quantum lifespan.** We now tie Theorem 1 together with the main result of [RZ2]. There we constructed a candidate solution of the IVP for the HCMA on a general projective Kähler manifold  $(M, \omega)$  by a quantization procedure, inspired by the formal analytic continuation argument of Semmes and Donaldson [D1, S2] and by the Phong-Sturm construction of geodesic segments [PS1] and test configurations geodesic rays [PS2] by finite-dimensional approximations. We first quantized the Hamiltonian flow determined by the initial velocity  $\dot{\varphi}_0$  and metric  $\omega_{\varphi_0} := \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$  as a semi-classical one-parameter subgroup of unitary Toeplitz complex Fourier integral operators

$$U_N(t) := \Pi_N e^{\sqrt{-1}tN\Pi_N\dot{\varphi}_0\Pi_N}\Pi_N$$

on the spaces  $H^0(M, L^N)$  of holomorphic sections of the quantizing line bundle  $L \rightarrow M$ . We then analytically continued the unitary group to an imaginary time semi-group

$$U_N(\sqrt{-1}s) : H^0(M, L^N) \rightarrow H^0(M, L^N). \quad (7)$$

We denote by  $U_N(-\sqrt{-1}s)(z, w)$  the Schwartz kernel of this operator with respect to the volume form  $(N\omega_{\varphi_0})^n$ . The potentials

$$\varphi_N(s, z) := \frac{1}{N} \log U_N(-\sqrt{-1}s, z, z) \quad (8)$$

are then readily seen to be subsolutions of the HCMA

$$\left\{ \begin{array}{ll} (\pi_2^* \omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0 & \text{on } S_T \times M, \\ \varphi(0, s, \cdot) = \varphi_0(\cdot) & \text{on } \{0\} \times \mathbb{R} \times M, \\ \frac{\partial \varphi}{\partial s}(0, s, \cdot) = \dot{\varphi}_0(\cdot) & \text{on } \{0\} \times \mathbb{R} \times M, \end{array} \right. \quad (9)$$

where  $\pi_2 : S_T \times M \rightarrow M$  is the projection (recall  $S_T = [0, T] \times \mathbb{R}$ ). We then defined the *quantum analytic continuation potential*  $\varphi_\infty$  by

$$\varphi_\infty(s, z) := \lim_{l \rightarrow \infty} (\sup_{N \geq l} \varphi_N)_{\text{reg}}(s, z), \quad (10)$$

where  $u_{\text{reg}}(z_0) := \lim_{\epsilon \rightarrow 0} \sup_{|z - z_0| < \epsilon} u(z)$  denotes the upper semi-continuous regularization of  $u$ . The *quantum lifespan* was then defined as

$$T_{\text{span}}^Q := \sup \{ T : \varphi_\infty \text{ solves (9) on } S_T \times M \}.$$

In the toric case, the Cauchy data  $\omega_{\varphi_0}, \dot{\varphi}_0$  gives rise to Cauchy data  $\psi_0, \dot{\psi}_0$  for (1), where  $\omega_{\varphi_0}$  equals  $\sqrt{-1}\partial\bar{\partial}\psi_0$  on the open orbit, and  $\dot{\psi}_0$  simply denotes the restriction of  $\dot{\varphi}_0$  to the open orbit. We proved in [RZ2] that the quantum analytic continuation potential coincides, on all of  $\mathbb{R}_+ \times \mathbb{R}^n$ , with the Legendre transform potential:

$$\psi_0 + \varphi_\infty(s, \cdot) = \psi_s$$

(by the symmetry,  $\varphi_\infty$  can be regarded as a function on  $\mathbb{R}_+ \times \mathbb{R}^n$ ). Theorem 1 therefore implies:

**COROLLARY 1.** *On a toric or Abelian variety,  $T_{\text{span}}^Q = T_{\text{span}}^{\text{cvx}}$ . In other words, the quantum analytic continuation potential  $\varphi_\infty$  given by (10) is a subsolution of the HCMA (9) that solves the Cauchy problem until  $T_{\text{span}}^{\text{cvx}}$ , and cease to solve it, even as a weak solution, after that time.*

As discussed in [RZ2], §1 and §§3.1, the question whether there exist alternative solutions of HRMA and HCMA is taken up in sequels to this article. In particular, in [RZ3] we give a characterization of the smooth lifespan  $T_{\text{span}}^\infty$  of the HCMA (9) from which we conclude that there exists no smooth solution to the HRMA (1) beyond the convex lifespan, namely  $T_{\text{span}}^{\text{cvx}} = T_{\text{span}}^\infty$ .

**1.2. Examples on  $S^2$ .** We illustrate with an example the behavior of the family of Kähler metrics (viewed as a path in  $\mathcal{H}_\omega$ ) associated with the quantum analytic continuation (or Legendre transform) potential.

Let  $(\mathbb{P}^1, \omega)$  denote the Riemann sphere equipped with a Kähler form  $\omega$ . Consider an  $S^1$ -invariant Kähler metric  $\omega_{\varphi_0}$  on  $\mathbb{P}^1$  that equals  $\sqrt{-1}\partial\bar{\partial}\psi_0$  away from the poles. Let  $\dot{\varphi}_0$  denote a given  $S^1$  invariant initial velocity. The geodesic equation

$$\frac{\partial^2 \psi}{\partial s^2} = \left( \frac{\partial^2 \psi}{\partial x^2} \right)^{-1} \left( \frac{\partial^2 \psi}{\partial x \partial s} \right)^2, \quad (11)$$

can be interpreted as the HRMA  $\det \nabla^2 \psi = 0$ . Denote

$$\psi'(s, x) := \frac{\partial \psi(s, x)}{\partial x}, \quad \dot{\psi}(s, x) := \frac{\partial \psi(s, x)}{\partial s}.$$

By letting  $y := \psi'(s, x)$ , and  $u(\psi'(s, x)) := x\psi'(s, x) - \psi(s, x)$ , a computation shows that equation (11) becomes  $\ddot{u}(s, y) = 0$ , solved by  $u(s, y) := u_0(y) + s\dot{u}_0(y)$ , with  $\dot{u}_0(y) = -\dot{\psi}_0((\psi'_0)^{-1}(y))$ .

Note that  $y = y_s, \theta$  are the action-angle variables for  $\omega_s = \sqrt{-1}\partial\bar{\partial}\psi(s, x)$ , i.e.  $\omega_s = dy_s \wedge d\theta$  and  $y_s$  is the moment map of the  $S^1$  action with respect to  $\omega_s$ . Let

$z = e^{x+\sqrt{-1}\theta}$  denote the holomorphic coordinate away from the poles. The metric  $g_s$  at time  $s$  is then given by

$$g_s = \psi''(s, x)(dx^2 + d\theta^2), \quad (12)$$

expressed in action-angle variables as

$$g_s = u(s)_{yy}dy^2 + \frac{1}{u(s)_{yy}}d\theta^2, \quad (13)$$

where  $u(s)_{yy}$  denotes the second derivative of the symplectic potential  $u$  at time  $s$  with respect to the action variable  $y = y_s$ . Also, let  $r_s$  denote the geodesic distance function from the north pole (the fixed point of the  $S^1$  action on which  $y_s$  takes its maximum). Then  $r_s$  is a function only of  $y_s$ . Hence the change of variables from  $y_s$  to  $r_s$  does not add any term containing  $d\theta$ , and

$$g_s = dr_s^2 + \frac{1}{u(s)_{yy}}d\theta^2. \quad (14)$$

Formulas (13)–(14) are valid only so long as  $u(s, \cdot) = u_0 + s\dot{u}_0$  remains convex, and they show that in that regime  $g_s$  is a smooth metric. The Legendre transform potential  $\psi$  defined by (4) provides an extension of the path of metrics  $\{g_s\}_{s \in [0, T_{\text{span}}^{\text{cvx}}]}$  given by (13) to  $s \in [T_{\text{span}}^{\text{cvx}}, \infty)$ , given by

$$g_s = u_s^{\star\star}(s, x)(dx^2 + d\theta^2). \quad (15)$$

Equation (4) also means that  $\psi(s, \cdot)$  is the Legendre dual of the convexification of  $u(s, \cdot)$ . The convexified symplectic potential  $u_s^{\star\star}$  is  $C^1$  but develops straight segments on intervals of non-convexity (see Figure 1). At these,  $u_s^{\star\star}(s)_{yy} = 0$ . The radial ‘height’ of the inverse image of this straight segment under  $y_s$  is zero but the lattice circle ( $S^1$  orbit) is of infinite radius. Thus, the metric develops an  $S^1$ -invariant delta-function singularity at the corresponding value of  $r_s$ . Theorem 1 is the statement that the path of metrics  $g_s$  ceases to be a geodesic precisely at  $T_{\text{span}}^{\text{cvx}}$ , when the singularities appear. However, it does satisfy the geodesic equation on a dense set, whose time slice is the complement of a discrete set of singular  $S^1$  orbits.

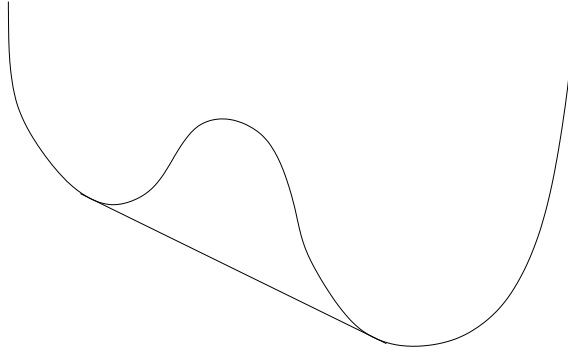


FIGURE 1. The graphs of  $u_s$  and  $u_s^{\star\star}$  over  $P$  for  $s > T_{\text{span}}^{\text{cvx}}$ .

**1.3. Asymptotic behavior of geodesic and subgeodesic rays.** It has been conjectured by Donaldson that smooth geodesic rays in the space of Kähler metrics should play an important role in questions regarding geometric ‘stability’ and existence of canonical metrics [D1]. An important question is therefore precisely which directions yield smooth geodesic rays. As mentioned in §§1.1, we prove in the sequel that for the HRMA (1) one has  $T_{\text{span}}^\infty = T_{\text{span}}^{\text{cvx}}$ . Therefore the directions of smooth toric geodesic rays are precisely those with infinite convex lifespan. Further, as a corollary of the results here and in the sequel we describe the limiting behavior of the rays obtained by solving the IVP using the quantum method of [RZ2], or, equivalently, by means of the Legendre transform method. This holds whether they be smooth geodesics or only ‘subgeodesics’, by which we mean subsolutions of the HCMA.

**1.4. Other Cauchy problems for the HRMA.** The IVP for geodesics in the space of toric metrics on toric varieties gives rise to a specific Cauchy problem for the HRMA. A natural question is to what extent do the techniques used in this article generalize to other Cauchy problems for the HRMA. We briefly discuss several possible problems and generalizations, some of which we hope to discuss in more detail elsewhere.

Aside from the fact that the Cauchy data and the Cauchy hypersurface are smooth, and that we require all solutions to be convex, the main distinctive features of the Cauchy problem (1) are:

- (i) the Cauchy hypersurface is an  $\mathbb{R}^n$ -slice of the total space  $[0, T] \times \mathbb{R}^n$ ,
- (ii) the initial convex function has linear growth at infinity, and the initial velocity is uniformly bounded.

An example of a Cauchy problem where (i), but not (ii), holds, is the IVP for geodesics in the space of toric metrics on Abelian varieties. In that setting the convex function  $\psi_0$  has quadratic growth at infinity. On the other hand, the gradient map  $\nabla\psi_0$  now maps to a torus instead of a polytope. It follows that the symplectic potential  $\psi_0^*$  can be considered as a convex function on  $\mathbb{R}^n$  with quadratic growth and no singularities. This fact eliminates the need for the analysis near the boundary of  $P$  that was necessary in this article. Although we do not go into the details here, using the methods of this series one may prove an analogue of our main results for this class of manifolds.

Note that the requirement in (ii) that the initial velocity be bounded, while the initial convex function divergent at infinity, guarantees that the gradient image of  $\psi_s$  remains constant, and this still held true in the Abelian case. Removing this assumption from (ii) would require dealing with a one-parameter family of gradient images  $P_s$ , and would certainly complicate some of the analysis.

Next, one may allow the Cauchy hypersurface to be more general, for example a smooth affine hypersurface in an affine manifold. Geometrically, this setting arises when one considers the IVP for geodesics in spaces of invariant metrics in various classes of manifolds with large symmetry groups (for examples see, e.g., [D2], §4). It would be interesting to obtain a formula for  $T_{\text{span}}^\infty$  analogous to the one for the

convex lifespan (3). We note that in the affine setting Foote [F1, F2] has given a sufficient condition on the Cauchy data and hypersurface to have  $T_{\text{span}}^\infty > 0$ .

Although the affine situation is certainly more complicated than the Euclidean one it seems plausible to us that one could generalize at least some of the techniques of the present series to the affine setting.

Finally, we mention that the Cauchy problem for the HRMA is also classically used to construct smooth Gauss flat hypersurfaces in  $\mathbb{R}^{n+2}$  (this is carried out in  $\mathbb{R}^3$  in [U]). It seems interesting to investigate what kind of singular hypersurfaces are obtained from the Legendre transform method when extended beyond the convex lifespan. A consequence of the a priori bound of Proposition 3 is that the resulting hypersurface has bounded total Gauss curvature, a fact that seems rather surprising. In §3 we briefly touch upon this point of view by proving Theorem 1 in the case  $n = 1$ , and also illustrate explicitly the finite curvature phenomenon.

## 2. BACKGROUND

In this section, we recall some basic definitions relating to convex analysis, the real Monge-Ampère operator, and the reduction of HCMA on toric Kähler manifolds to the HRMA. For additional necessary notation and definitions that are used throughout we refer to [RZ2], §§4.3.

**2.1. Convex analysis.** For general background on Legendre duality and convexity we refer the reader to [HL1, HL2, Ro], whose notation and terminology we largely adhere to. Denote by

$$\text{co } A$$

the convex hull of the set  $A$ . Given a function  $f$  defined on a set  $P \subset \mathbb{R}^n$  let

$$\text{epi } f := \{(x, r) : r \geq f(x), x \in P\}$$

denote its epigraph. A vector  $v \in (\mathbb{R}^n)^*$  is said to be a subgradient of a function  $f$  at a point  $x$  if  $f(z) \geq f(x) + \langle v, z - x \rangle$  for all  $z$ . The set of all subgradients of  $f$  at  $x$  is called the subdifferential of  $f$  at  $x$ , denoted  $\partial f(x)$ .

The Legendre-Fenchel conjugate of a continuous function  $f = f(x)$  on  $\mathbb{R}^n$  is defined by [Fe]

$$f^*(y) := \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - f(x)).$$

For simplicity, we will refer to  $f^*$  sometimes as the Legendre dual, or just dual, of  $f$ .

**DEFINITION 2.1.** *A convex function  $f$  is called proper if it is not identically  $+\infty$  and is uniformly bounded below. A proper convex function is called closed if it is lower semi-continuous. The domain of  $f$  is defined by  $\text{dom}(\partial f) := \{x : \partial f(x) \neq \emptyset\}$ .*

- (i) *A function on a convex set  $C$  is called strictly convex on  $C$  if for every  $\lambda \in (0, 1)$  and all distinct points  $x_1, x_2 \in C$  holds  $f((1 - \lambda)x_1 + \lambda x_2) < (1 - \lambda)f(x_1) + \lambda f(x_2)$ .*
- (ii) *A proper convex function is called essentially strictly convex if it is strictly convex on every convex subset of  $\text{dom}(\partial f)$ .*
- (iii) *A proper convex function is called essentially smooth if  $C := \text{int}(\text{dom}(\partial f)) \neq \emptyset$ ,*

if  $f$  is differentiable on  $C$  and if  $\lim_{i \rightarrow \infty} |\nabla f(x_i)| = +\infty$  whenever  $\{x_i\}_{i \geq 1}$  is a sequence in  $C$  converging to  $x \in \partial C$ .

**2.2. The real Monge-Ampère operator.** We recall the definition of the Monge-Ampère operator and its basic characterization, due to Alexandrov and Rauch-Taylor. Let  $M(\mathbb{R}^{n+1})$  denote the space of differential forms of degree  $n+1$  on  $\mathbb{R}^{n+1}$  whose coefficients are Borel measures (i.e., currents of degree  $n+1$  and order 0).

**PROPOSITION 2.2.** (See [RT], Proposition 3.1) *Define by*

$$\text{MA}f := d \frac{\partial f}{\partial x^1} \wedge \cdots \wedge d \frac{\partial f}{\partial x^{n+1}},$$

*an operator  $\text{MA} : C^2(\mathbb{R}^{n+1}) \rightarrow M(\mathbb{R}^{n+1})$ . Then  $\text{MA}$  has a unique extension to a continuous operator on the cone of convex functions.*

An alternative, geometric, definition is due to Alexandrov, and uses the notion of a subdifferential of a convex function.

**PROPOSITION 2.3.** (See [RT], Section 2) *For any convex function  $f$ , the measure  $\text{MA}f$ , defined by*

$$(\text{MA}f)(E) := \text{Lebesgue measure of } \partial f(E),$$

*is a Borel measure.*

The following result of Rauch-Taylor links these two definitions and will be crucial below.

**THEOREM 2.4.** (See [RT], Proposition 3.4) *For every convex function  $f$  on  $\mathbb{R}^{n+1}$  one has the equality of Borel measures  $\text{MA}f = \text{MA}f$ . In particular, the real Monge-Ampère measure is zero if and only if the image of the subdifferential map has Lebesgue measure zero in  $\mathbb{R}^{n+1}$ .*

**2.3. The Cauchy problem for the symplectic potential.** It is well-known that the Legendre transform linearizes the HRMA. This fact also has a geometric interpretation that we now briefly review.

Let  $(M, \omega)$  be a toric Kähler manifold of complex dimension  $n$  and let  $\mathbf{T} = (S^1)^n$  denote the real torus of dimension  $n$  which acts on  $(M, \omega)$  in a Hamiltonian fashion. We denote by  $\mathcal{H}(\mathbf{T})$  the class of  $\mathbf{T}$ -invariant Kähler metrics in the cohomology class of  $\omega$ . On the open-orbit of  $\mathbf{T}^{\mathbb{C}} = (\mathbb{C}^*)^n$ , a  $\mathbf{T}$ -invariant Kähler metric has a Kähler potential  $\psi$  and we also write  $\psi \in \mathcal{H}(\mathbf{T})$ .

Since it is  $\mathbf{T}$ -invariant, the Kähler potential may be identified with a smooth strictly convex function on  $\mathbb{R}^n$  in logarithmic coordinates. Therefore its gradient  $\nabla \psi$  is one-to-one onto  $P = \overline{\text{Im} \nabla \psi}$  and one has the following explicit expression for its Legendre dual ([Ro], or [R], p. 84–87),

$$u(y) = \psi^*(y) = \langle y, (\nabla \psi)^{-1}(y) \rangle - \psi \circ (\nabla \psi)^{-1}(y), \quad (16)$$

which is a smooth strictly convex function on  $P$ , satisfying

$$\nabla u(y) = (\nabla \psi)^{-1}(y), \quad (17)$$

and

$$(\nabla^2 u(y))^{-1} = \nabla^2 \psi((\nabla \psi)^{-1}(y)). \quad (18)$$

Following Guillemin, the function  $u$  is called the symplectic potential of the metric  $\sqrt{-1}\partial\bar{\partial}\psi$ . The space of all symplectic potentials is denoted by  $\mathcal{LH}(\mathbf{T})$ . Put

$$u_G := \sum_{k=1}^d l_k \log l_k. \quad (19)$$

A result of Guillemin [G1] states that for any symplectic potential  $u$  the difference  $u - u_G$  is a smooth function on  $P$  (that is, up to the boundary). In other words,

$$\mathcal{LH}(\mathbf{T}) = \{u \in C^\infty(P \setminus \partial P) : u = u_G + F, \text{ with } F \in C^\infty(P)\}. \quad (20)$$

The Legendre transform is an isometry between  $(\mathcal{H}(\mathbf{T}), g_{L^2})$  and  $(\mathcal{LH}(\mathbf{T}), L^2(P))$ . It transforms the Christoffel symbols of  $(\mathcal{H}(\mathbf{T}), g_{L^2})$  to zero and thus linearizes the Monge-Ampère equation to the equation  $\ddot{u} = 0$  (for more on this see [Gu, S1], or [RZ1], §3). The differential of the Legendre transform acts as minus the identity, that is if  $\eta_s$  is a curve in  $\mathcal{H}(\mathbf{T})$  and if  $u_s := \eta_s^*$  are the corresponding symplectic potentials then

$$\dot{\eta}_s = -\dot{u}_s \circ \nabla \eta_s \quad (21)$$

(see, e.g., [R], p. 85). Therefore the IVP on  $(\mathcal{H}(\mathbf{T}), g_{L^2})$  is transformed to the following initial value problem for geodesics in the space of symplectic potentials:

$$\ddot{u} = 0, \quad u_0 = \psi_0^*, \quad \dot{u}_0 = -\dot{\psi}_0 \circ (\nabla \psi_0)^{-1}. \quad (22)$$

### 3. LEGENDRE CONTINUATION OF FLAT SURFACES IN $\mathbb{R}^3$

Our purpose in this section is to explain the proof of Theorem 1 in the case where  $n = 1$ , i.e., for the Cauchy problem for the 2-dimensional HRMA of §§1.2. This special setting is simpler to visualize explicitly and helps motivate and better capture some of the constructions carried out in the proof of Theorem 1, and also, part of the analysis simplifies. Many of the assertions in this section are stated without rigorous justification, and their proofs can be found (for all  $n$ ) in later sections.

The graph of  $\psi(s, x)$

$$\{(s, x, \psi(s, x))\} \subset \mathbb{R}^3$$

over  $[0, T_{\text{span}}^{\text{cvx}}] \times \mathbb{R}$  is a flat surface since its second fundamental form is proportional to the Hessian of  $\psi$ . We are interested in what happens to this surface when extended beyond  $T_{\text{span}}^{\text{cvx}}$ : does  $\psi$  still define a flat surface, in a weak sense, after  $T_{\text{span}}^{\text{cvx}}$ ?

Let  $\omega_{\text{FS}}$  denote the Fubini-Study form of constant Ricci curvature 1 on the Riemann sphere, given locally by

$$\omega_{\text{FS}} = \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

The associated open-orbit Kähler potential can be taken as

$$\psi_0(x) = \log(1 + |z|^2) - \frac{1}{2}\operatorname{Re} z = \log(1 + e^{2x}) - x. \quad (23)$$

The corresponding moment polytope is  $[-1, 1]$ , and the symplectic potential dual to  $\psi_0$  can be computed via the moment map  $y(x) = \psi'_0(x) \in [-1, 1]$ ,

$$u_0(y) = (1 + y)\log(1 + y) + (1 - y)\log(1 - y), \quad y \in [-1, 1]. \quad (24)$$

Let  $\dot{\varphi}_0 \in C^\infty(S^2)$  be given and set

$$\dot{u}_0 = -\dot{\varphi}_0((\psi'_0)^{-1}(\cdot)) = -\dot{\psi}_0((\psi'_0)^{-1}(\cdot)) \in C^\infty([-1, 1]).$$

We start with the analysis of  $u_s^{**}$ . By (24) we have  $\lim_{y \rightarrow \pm 1} u'_0 = \pm\infty$ , and the same holds for  $u_s$  since  $\dot{u}_0$  is bounded on  $P$ . Therefore, when restricted to a neighborhood of  $\{\pm 1\}$  in  $[-1, 1]$  the tangent lines to the graph of  $u_s$  lie below the graph. Hence,  $A_s \cap \{\pm 1\} = \emptyset$ , and the graphs of  $u_s$  and of  $u_s^{**}$  differ only above  $(-1 + \epsilon, 1 - \epsilon)$  for some  $\epsilon > 0$ . Above every connected component of  $A_s$  the graph of  $u_s^{**}$  is necessarily affine with slope precisely equal to the derivative of  $u_s$  at the end-points. Hence  $u_s^{**}$  is continuously differentiable.

Let  $s > T_{\text{span}}^{\text{cvx}}$  so  $A_s \neq \emptyset$ . Let us assume for simplicity that  $A_s$  is connected, with  $A_s = (a_s, b_s) \subset [-1, 1]$ , and set  $x_s := \nabla u_s^{**}((a_s, b_s))$ . Then the graph of  $\psi_s$  has a corner of angle  $\alpha_s = \tan^{-1} a_s - \tan^{-1} b_s$  at  $(x_s, \psi_s(x_s))$ , and is smooth elsewhere. The set of reachable subgradients of  $\psi_s$  at  $x_s$  is  $\gamma\psi_s(x_s) = \{a_s, b_s\}$ . It follows that  $\gamma\psi(s, x_s)$ , the set of reachable subgradients of  $\psi$  at  $(s, x_s)$ , contains  $\{(-\dot{u}_0(a_s), a_s), (-\dot{u}_0(b_s), b_s)\}$ . Thus, by convexity, the total subdifferential of  $\psi$  at  $(s, x_s)$  contains the line connecting these two points:

$$\operatorname{co}\{(-\dot{u}_0(a_s), a_s), (-\dot{u}_0(b_s), b_s)\} \subset \partial\psi(s, x_s). \quad (25)$$

On the other hand, the graph of  $-\dot{u}_0$  cannot be linear even locally on  $(a_s, b_s)$  since that would imply that  $u_0 + s\dot{u}_0$  were strictly convex on some part of  $(a_s, b_s)$ . It follows that the affine segment in (25) intersects the graph of  $-\dot{u}_0$  over  $(a_s, b_s)$  only in a discrete set of points. Since  $A_s$  is monotonically increasing in  $s$  in a continuous fashion, we conclude that the image of  $\partial\psi$  fills-in a region in  $\mathbb{R} \times [-1, 1]$  above the graph of  $-\dot{u}_0$ , and consequently that  $\psi$  is not a weak solution of the 2-dimensional HRMA on  $[0, T] \times \mathbb{R}$  for any  $T > T_{\text{span}}^{\text{cvx}}$ . Moreover, the region filled out, as  $T$  tends to infinity, is precisely the region above the graph of  $-\dot{u}_0$  and below the graph of  $-(\dot{u}_0)^{**}$ . We conclude that the non-compact surface defined by  $\psi$  is flat precisely on the complement of the corner set, but not globally flat. However, its total

curvature in the sense of Alexandrov while positive, is finite, and is concentrated on the codimension one corner set.

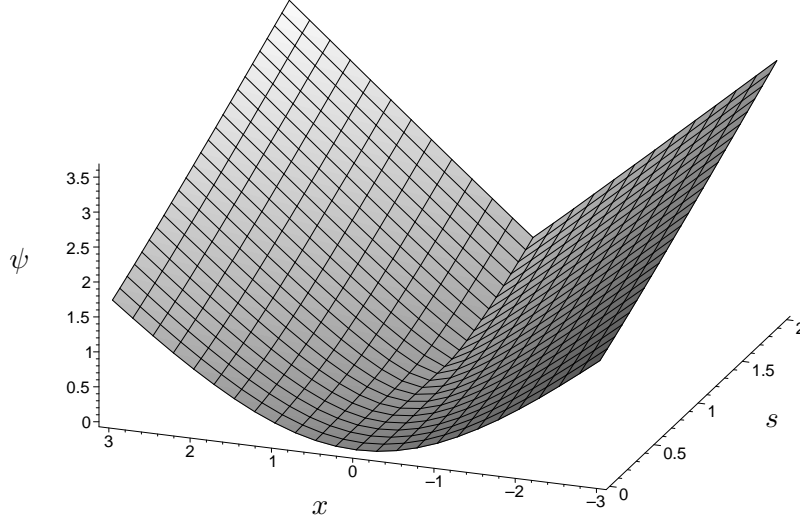


FIGURE 2. The graph of  $\psi$  for  $(s, x) \in [0, 2] \times [-3, 3]$  and Cauchy data  $(\psi_0, \dot{\psi}_0)$  given by (23) and (26).

**EXAMPLE 3.1.** The graph of  $\psi$  may be parametrized with respect to the moment coordinate on the domain of its invertibility, namely,

$$\text{graph of } \psi = \left\{ (s, \nabla u_s(y), \langle y, \nabla u_s(y) \rangle - u_s(y)) : s \in \mathbb{R}_+, y \in P \setminus (A_s \cup \partial P) \right\}.$$

To visualize the graph of  $\psi$  in a specific example let  $\psi_0$  be given by (23), and set

$$\dot{\psi}_0(z) = \left( \frac{|z|^2 - 1}{|z|^2 + 1} \right)^2, \quad \text{or,} \quad \dot{\psi}_0(x) = \left( \frac{e^{2x} - 1}{e^{2x} + 1} \right)^2. \quad (26)$$

Hence,  $\dot{u}_0(y) = -y^2$ .

A portion of this graph is drawn in Figure 2. In this example,  $T_{\text{span}}^{\text{cvx}}(\psi_0, \dot{\psi}_0) = 1$ , and the graph of  $\psi$  develops a kink along the corner set above  $\Sigma_{\text{sing}} = \{x = 0, s > 1\}$ . For each  $s > 1$  the graph of  $\psi_s$  has a kink at  $x = 0$ , and so the corner set of  $\psi$  is precisely the union over  $s$  of the corner sets of  $\psi_s$ . Moreover, in this special example  $u_0$  and  $\dot{u}_0$  are even functions. The same is then true for  $\psi_s$ , and one may further show that  $\psi$  is differentiable in  $s$ . Let  $A_s = (-a_s, a_s)$ . Then  $\partial_x \psi(s, x_s) = [-a_s, a_s]$ ,

and

$$\partial\psi(s, x_s) = \{\dot{\psi}_s(x_s)\} \times \partial_x\psi(s, x_s) = \{(a_s^2, y) : y \in [-a_s, a_s]\}.$$

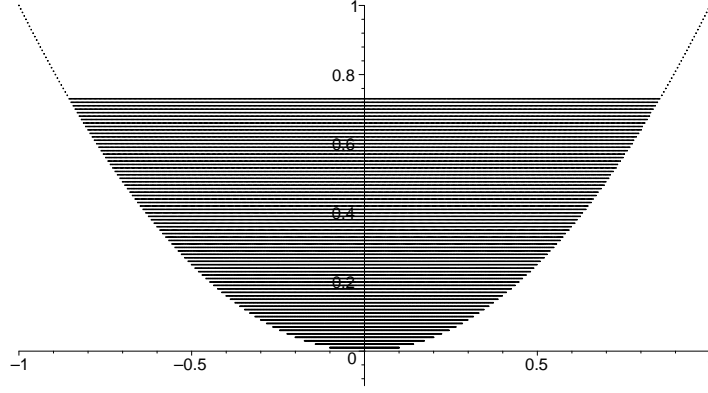


FIGURE 3. The dotted and shaded region is  $\partial\psi([0, 1.5] \times \mathbb{R})$  for the Cauchy data  $(\psi_0, \dot{\psi}_0)$  given by (23) and (26).

A typical set of the form  $\partial\psi([0, T] \times \mathbb{R}^n)$  is illustrated in Figure 3. It is contained in  $\text{epi}(-\dot{u}_0) \setminus \text{epi}(-(\dot{u}_0)^{\star\star})$ , with  $(\dot{u}_0)^{\star\star}(y) \equiv -1$ . The total curvature of  $\text{graph}(\psi)$  is bounded from above by the area of the difference between the epigraph of  $-\dot{u}_0$  and that of  $-(\dot{u}_0)^{\star\star}$  (over  $[-1, 1]$ ),

$$\int_0^\infty \int_{-\infty}^\infty \text{MA } \psi = \int_{-1}^1 (1 - y^2) dy = \frac{4}{3}.$$

Indeed, the total curvature is given by the measure in  $S^n$  of the unit normals to the surface, that is dominated from above by the Lebesgue measure of its stereographic projection in  $\mathbb{R}^n$ . The latter is precisely the Monge-Ampère mass (see [RT], Proposition 2.3, or [P]). Thus, while the surface has an infinitely long corner set, the surface quickly becomes ‘almost linear’ outside it, and in a manner guaranteeing its total curvature will remain finite.

#### 4. INTERIOR $C^1$ REGULARITY OF $u_s^{\star\star}$

The purpose of this section is to prove Lemma 4.1, which shows that  $u_s^{\star\star}$  is  $C^1$  on the interior of  $P$ .

Let  $f \in C^0(P)$ . The biconjugate  $f^{**}$  of  $f$  is a convex function on  $P$  and can be characterized as the ‘convex envelope’ of  $f$  ([HL2], Theorem 1.3.5, p. 45)

$$\begin{aligned} f^{**}(x) &= \sup \{g(x) : g \text{ convex on } P \text{ and } g \leq f\} \\ &= \sup \{a(x) : a \text{ affine on } P \text{ and } a \leq f\}. \end{aligned} \quad (27)$$

Consider a smooth function defined on a compact convex set and smooth in its interior. It is not true in general that its biconjugate is differentiable in the interior. However, in our situation the biconjugate enjoys the maximal degree of regularity possible in general, i.e., it is  $C^{1,1}$  (no gain is achieved by considering a real-analytic function). General results in the literature (see [BH, GR, KK] and [HL2], §X.1.5) are usually stated for functions defined on all of  $\mathbb{R}^n$  that obey certain growth conditions at infinity (cf. [GR], (2.3), [HL2], (1.5.2), p. 50, [BH], (18), [KK], (1)) or else make certain assumptions regarding boundary behavior that need not hold in our situation (cf. [GR], (4.1) and particularly Remark 4.3). Therefore, and also since the constructions involved will be useful later, we find it beneficial to extract from the references above partially self-contained proofs of differentiability of  $u_s^{**}$ . The  $C^{1,1}$  estimate can be deduced from the  $C^1$  estimate [RZ3].

**LEMMA 4.1.** *Let  $s > 0$ .*

(i) *The graph of the Legendre double dual of  $u_s$  is the lower boundary of the closed convex hull of the epigraph of  $u_s$ .*

(ii)  *$u_s^{**} \in C^1(P \setminus \partial P) \cap C^0(P)$ .*

*Proof.* (i) From (27) it follows that for any finite convex function  $f$  defined on  $P$ , one has

$$\overline{\text{co epi } f} = \text{epi } f^{**}, \quad (28)$$

Since  $u_s$  is smooth and continuous up to the boundary of  $P$  the convex hull of  $\text{epi } u_s$  is closed, and the result follows.

(ii) From (27) we have  $u_s^{**} \leq u_s$ . Since  $u_s$  majorizes a linear function on  $P$  it also follows that  $u_s^{**}$  is bounded below. Hence  $u_s^{**} \in C^0(P)$ .

We divide the proof of the  $C^1$  estimate into two steps and closely follow [BH, HL2]. The proof will show that for any compact subset  $\Omega \subset P \setminus \partial P$  there exists a compact subset  $\Omega' \subset P \setminus \partial P$  with  $\Omega \subset \Omega'$  such that

$$\|\nabla u_s^{**}\|_{C^0(\Omega)} \leq \|\nabla u_s\|_{C^0(\Omega')} < C(s+1),$$

with  $C = C(\Omega, \Omega', u_0, \dot{u}_0) > 0$ .

*First step.* Denote by  $\Delta_{n+1} \subset \mathbb{R}^{n+1}$  the unit simplex

$$\Delta_{n+1} := \left\{ \lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1} : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

Recall the following representation formula for the biconjugate function ([Ro], Corollary 17.1.5),

$$u_s^{**}(y) = \inf \left\{ \lambda \cdot (u_s(y_1), \dots, u_s(y_{n+1})) : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\}. \quad (29)$$

We claim that there exist—for each  $y \in P \setminus \partial P$ —points  $y_1, \dots, y_{n+1} \in P$  and a vector  $\lambda \in \Delta_{n+1}$  such that

$$(y, u_s^{\star\star}(y)) = \sum_{i=1}^{n+1} \lambda_i (y_i, u_s(y_i)). \quad (30)$$

To see that, observe that according to (i) the epigraph of  $u_s^{\star\star}$  is the closed convex hull of the epigraph of  $u_s$ . Hence, by convexity, for each  $y \in P \setminus \partial P$  there exists an  $m \in \mathbb{N}$  and a  $\lambda \in \Delta_m$  and  $\{(y_i, r_i)\}_{i=1}^m \subset \text{epi } u_s$  such that

$$(y, u_s^{\star\star}(y)) = \sum_{i=1}^m \lambda_i (y_i, r_i). \quad (31)$$

But then since  $u_s(y_i) \leq r_i$ , and  $\sum_{i=1}^m \lambda_i (y_i, u_s(y_i))$  also belongs to  $\text{epi } u_s$  it follows that whenever  $\lambda_i > 0$  there holds  $r_i = u_s(y_i)$ : otherwise  $(y, u_s^{\star\star}(y))$  would lie “directly above” another point in the convex hull of  $\text{epi } u_s$ , contradicting (i). Finally, using (i), since the convex hull of the epigraph of  $u_s$  is closed, it follows that  $(y, u_s^{\star\star}(y))$  lies on its boundary; then by a consequence of Carathéodory’s theorem it is possible to take  $m = n + 1$  in (31) ([HL1], Proposition 4.2.3, p. 126).

*Second step.* When equation (30) holds, set

$$I := \{i : \lambda_i > 0\}. \quad (32)$$

Following the terminology of [BH], §3, we say that the points  $\{y_i\}_{i \in I}$  are *called upon by*  $y$ . We omit from the notation the dependence of  $y_i$  on  $y$ . By (27)  $u_s^{\star\star}$  is convex and  $u_s^{\star\star} \leq u_s$ . Hence,

$$u_s^{\star\star}(y) \leq \lambda \cdot (u_s^{\star\star}(y_1), \dots, u_s^{\star\star}(y_{n+1})) \leq \lambda \cdot (u_s(y_1), \dots, u_s(y_{n+1})),$$

which together with (30) implies

$$u_s^{\star\star}(y_i) = u_s(y_i), \quad \forall i \in I. \quad (33)$$

Note that whenever  $g \leq f$  and  $g(x) = f(x)$  for  $g$  convex and  $f$  differentiable, then  $g$  is differentiable at  $x$  and  $\nabla g(x) = \nabla f(x)$  (indeed any element of  $\partial g(x)$  defines a supporting hyperplane to  $f$  at  $x$ , hence a tangent hyperplane).

**CLAIM 4.2.** *Let  $y \in P \setminus \partial P$ , and let  $\{y_i\}$  and  $I$  be defined by (30) and (32), respectively. Then for each  $i \in I$  one has  $y_i \in P \setminus \partial P$ .*

*Proof.* The proof relies on the formula

$$\partial u_s^{\star\star}(y) = \bigcap_{i \in I} \partial u_s(y_i), \quad (34)$$

([BH], Theorem 3.6, [HL2], Theorem 1.5.6, p. 53) whose derivation applies verbatim in our situation: indeed  $y^* \in \partial u_s^{\star\star}(y)$  iff  $u_s^{\star\star}(y^*) + u_s^{\star\star}(y) = \langle y^*, y \rangle$ , or (using  $u_s^{\star\star} = u_s^*$  and (30))  $\sum_{i \in I} \lambda_i (u_s^*(y^*) + u_s(y_i) - \langle y^*, y_i \rangle) = 0$ , and (34) follows by the characterization of the subdifferential.

Since  $u_s^{\star\star} \in C^0(P)$  and  $y \in P \setminus \partial P$ , there exists a neighborhood of  $y$  in  $P \setminus \partial P$  on which  $u_s^{\star\star}$  is finite. Hence, since  $u_s^{\star\star}$  is convex on  $P$  it follows that the left hand side of (34) is nonempty (observe that this argument fails on the boundary: formally

one thinks of  $u_s$  as defined on all of  $\mathbb{R}^n$  and identically equal to  $+\infty$  outside  $P$ ). However, by Guillemin's formula (19) the subdifferential of  $u_G$  and hence of  $u_s$  at every point of  $\partial P$  is empty. Therefore (34) implies that  $y_i \in P \setminus \partial P$ .  $\square$

Since  $u_s \in C^1(P \setminus \partial P)$  it follows from the Claim and (34) that  $u_s^{**}$  is differentiable at  $y_i$ ,  $i \in I$ , and  $\nabla u_s^{**}(y_i) = \nabla u_s(y_i)$  (observe that by (33)  $y_i$  is the only point called upon by  $y_i$  and apply (34) to  $y = y_i$ ). Therefore, by using (34) once more, it follows that  $\partial u_s(y_i) = \{\nabla u_s(y_i)\}$  is identical for all  $i \in I$ . We conclude

$$\nabla u_s^{**}(y) = \nabla u_s^{**}(y_i) = \nabla u_s(y_i), \quad (35)$$

hence  $u_s^{**}$  is differentiable at  $y$ , as desired. Continuity of  $\nabla u_s^{**}$  now follows from convexity ([Ro], Corollary 25.5.1, [KK]). This concludes the proof of Lemma 4.1.  $\square$

## 5. THE SET $A_s$

As indicated in §1 the following set plays an important role in the analysis:

$$A_s := \{y \in P : u_s(y) \neq u_s^{**}(y)\}. \quad (36)$$

In this section, we first prove a basic characterization of this set which is later used repeatedly. In addition, we give a geometric description of this set that is used to prove Lemma 6.1 in the next section. In §8 we prove further properties of  $A_s$  which are needed for the proof of Theorem 1.

When  $s \leq T_{\text{span}}^{\text{cvx}}$  the set  $A_s$  is empty. When  $s > T_{\text{span}}^{\text{cvx}}$  this set is an open non-empty set relative to the topology of  $P$  (it may intersect  $\partial P$ —see Example 5.2 below). The set  $A_s \cap (P \setminus \partial P)$  is a non-empty open set in the usual topology of  $\mathbb{R}^n$ . Both of these assertions follow from the continuity of  $u_s$  and  $u_s^{**}$ .

**LEMMA 5.1.** *One has*

$$A_s \cap P \setminus \partial P = \{y \in P \setminus \partial P : \partial u_s(y) = \emptyset\}.$$

*Proof.* Let  $y \in P \setminus \partial P$ . If  $\partial u_s(y) \neq \emptyset$  then  $u_s^{**}(y) = u_s(y)$  (by definition  $u_s^{**}$  majorizes the supporting affine function at  $y$ , while  $u_s^{**} \leq u_s$  always), hence

$$A_s \cap P \setminus \partial P \subset \{y \in P \setminus \partial P : \partial u_s(y) = \emptyset\}.$$

Conversely, if  $u_s(y) = u_s^{**}(y)$  then  $y$  itself can be viewed as the only point called upon by  $y$ . According to Lemma 4.1, convexity of  $u_s^{**}$ , Equation (34), and since  $y \in P \setminus \partial P$ , we have

$$\partial u_s(y) = \partial u_s^{**}(y) = \{\nabla u_s(y)\} = \{\nabla u_s^{**}(y)\} \neq \emptyset. \quad (37)$$

Hence  $\{y \in P \setminus \partial P : \partial u_s(y) = \emptyset\} \subset A_s \cap P \setminus \partial P$ .  $\square$

It is important to note that unlike the intuition from the case  $n = 1$  (see §3), the set  $A_s$  may intersect the boundary of  $P$ . We illustrate with a simple example.

**EXAMPLE 5.2.** We follow the notation of §3. Consider  $M = \mathbb{P}^1 \times \mathbb{P}^1$  endowed with the Guillemin Kähler structure  $\omega_{\varphi_0} = \pi_1^* \omega_{\text{FS}} + \pi_2^* \omega_{\text{FS}}$ , where  $\pi_j : M \rightarrow \mathbb{P}^1$  is the projection onto the  $j$ -th factor. The associated polytope is  $P = [-1, 1] \times [-1, 1]$ , and the symplectic potential dual to  $\psi_0$  is

$$u_0(y) = \frac{1}{2} \sum_{j=1}^2 (1 + y_j) \log(1 + y_j) + (1 - y_j) \log(1 - y_j).$$

Let  $f(y) : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly concave function, and let

$$\dot{u}_0(y) = f(y_1), \quad y \in P.$$

Then for  $s$  sufficiently large,

$$A_s \cap \{(y_1, \pm 1) : y_1 \in (-1, 1)\} \neq \emptyset.$$

Nevertheless, as illustrated in this example, what does generalize from the case  $n = 1$  is the following fact.

**LEMMA 5.3.** *The set  $A_s$  is at positive distance from the vertices of  $P$ .*

*Proof.* Recall that by the Delzant condition, if  $p$  is a vertex of  $P$  then  $p$  is the intersection of exactly  $n$  of the  $d$  defining half-spaces of  $P$ ; in the notation of [RZ2], §§4.2,

$$\{p\} = \bigcap_{k=1}^n \{y \in \mathbb{R}^n : l_{j_k(p)}(y) = 0\}, \quad (38)$$

with  $j_k \in \{1, \dots, d\}$ . The functions  $l_{j_1(p)}(y), \dots, l_{j_n(p)}(y)$  provide a coordinate system in which the Hessian of  $u_G$  is diagonal with eigenvalues  $\{\lambda_k = f_k l_{j_k(p)}^{-1} + g_k\}_{k=1}^n$ , where  $f_k, g_k \in C^\infty(P)$  are bounded functions up to the boundary, and  $f_k > 0$ . The same holds for  $u_0 \in \mathcal{LH}(T)$  and hence also for  $u_s$  since  $u_s - u_0$  is smooth up to the boundary. Hence, for some  $\epsilon > 0$ , that we assume is the largest such possible,  $u_s$  is strictly convex on  $P \cap B(p, \epsilon)$ , where  $B(p, \epsilon) := \{v \in \mathbb{R}^n : |v - p| < \epsilon\}$ .

Let now  $\delta \in (0, \epsilon)$  and let  $y \in B(p, \delta) \cap P \setminus \partial P$ . By convexity the graph of the tangent hyperplane  $H(w) := u_s(y) + \langle w - y, \nabla u_s(y) \rangle$  to the graph of  $u_s$  at  $y$  supports the graph of  $u_s$  above  $P \cap B(p, \epsilon)$ . Suppose that this is not true globally, namely,  $\min_P u_s - H < 0$ . From the explicit expression for  $u_G$  we obtain that in the aforementioned coordinates the gradient of  $u_s$  can be written in the form  $(\log l_{j_1(p)} + h_1, \dots, \log l_{j_n(p)} + h_n)$ , where  $h_j \in C^\infty(P)$ ,  $j = 1, \dots, n$ . Since  $u_s$  is bounded on  $P$ , it follows that by making  $\delta > 0$  smaller (and hence making  $y$  closer to  $p$ ), we may assume that for some  $w \in P \setminus (\{y\} \cup \partial P)$ , the graph of  $H$  is tangent to that of  $u_s$  at  $w$ , namely  $\nabla u_s(y) = \nabla u_s(w)$ . However, we already know that  $w \notin B(p, \epsilon)$ , and by evaluating the expression for  $\nabla u_s$  with respect to the coordinates  $l_{j_1(p)}, \dots, l_{j_n(p)}$  at  $w$  and comparing with (38) we obtain a contradiction once  $\delta$  is chosen sufficiently small with respect to  $\epsilon$  (in a manner depending on  $\max_j \max_P h_j$ ). It follows that for such a choice of  $\delta > 0$  (that depends only on  $p$  and  $s$ ) we have  $(B(p, \delta) \cap P) \cap A_s = \emptyset$ .  $\square$

6. ESSENTIAL SMOOTHNESS OF  $u_s^{\star\star}$ 

Having established the interior  $C^1$  regularity of  $u_s^{\star\star}$  we now turn to a result concerning the boundary behavior of its gradient.

**LEMMA 6.1.** *For each  $s > 0$  the function  $u_s^{\star\star}$  is essentially smooth.*

*Proof.* First observe that by Lemma 4.1 (i) the function  $u_s^{\star\star}$  is proper, as required by Definition 2.1. Next, observe that  $u_s^{\star\star}$  is differentiable on  $P \setminus \partial P = \text{int dom}(\partial u_s^{\star\star})$  by Lemma 4.1 (ii). Also, from Lemmas 4.1 and 5.3 and their proofs, if  $\{w_i\} \subset P \setminus \partial P$  is a sequence converging to one of the vertices of  $P$  then  $\lim_{i \rightarrow \infty} |\nabla u_s^{\star\star}(w_i)| = +\infty$  (since  $\nabla u_s^{\star\star}(w_i) = \nabla u_s(w_i)$  for large  $i$ ). Consider now a sequence  $\{w_i\} \subset P \setminus \partial P$  converging to a point  $p$  in  $\partial P$  contained in the interior of a face  $F$  of dimension 1, that we assume, without loss generality, is cut out by the equations  $l_j = 0$ , for  $j = 1, \dots, n-1$ , with  $l_n \in [0, C]$  a coordinate on this face. Then  $\lim_{i \rightarrow \infty} l_j(w_i) = 0$  for  $j = 1, \dots, n-1$ . Let  $\{p_i^0\}, \{p_i^1\} \subset P \setminus \partial P$  be sequences defined by the equations  $l_k(p_i^j) = l_k(w_i)$ ,  $k = 1, \dots, n-1$  and  $l_n(p_i^j) = Cj + (1/2 - j)\epsilon/i$ , with  $j = 0, 1$ , where  $\epsilon > 0$  is some sufficiently small constant depending on  $p$  and  $P$ . Then  $\lim_{i \rightarrow \infty} p_i^j$ , with  $j = 0, 1$ , are two distinct vertices of  $P$ . Using the functions  $l_1, \dots, l_n$  as coordinates in a neighborhood of the face  $F \subset P$ , it follows from Guillemin's formula (19) that for each  $k \in \{1, \dots, n-1\}$  one has

$$L_k := \lim_{i \rightarrow \infty} \frac{\partial u_s^{\star\star}}{\partial l_k}(p_i^0) = \lim_{i \rightarrow \infty} \frac{\partial u_s^{\star\star}}{\partial l_k}(p_i^1) \in \{\pm\infty\},$$

since by Lemma 5.3 one may replace  $u_s^{\star\star}$  by  $u_s$  in this equation. Considering then the function  $u_s^{\star\star}$  restricted to the line connecting  $p_i^0$  and  $p_i^1$  (necessarily contained in  $P$  by convexity), and taking the limit it follows that

$$\lim_{i \rightarrow \infty} \frac{\partial u_s^{\star\star}}{\partial l_k}(w_i) = L_k,$$

and so  $\lim_{i \rightarrow \infty} |\nabla u_s^{\star\star}(w_i)| = +\infty$ . The general case where  $p$  is contained in a boundary face of dimension  $m \leq n-1$  now follows by induction on  $m$ , using arguments as above.  $\square$

7. GRADIENT AND SUBDIFFERENTIAL MAPPINGS OF  $u_s^{\star\star}$ 

The surjectivity of  $\nabla u_s^{\star\star}$  can be proved as follows. By definition,

$$\psi_s(x) = \sup_{y \in P} (\langle x, y \rangle - u_s(y)),$$

and since  $P$  is compact and  $u_s$  bounded it follows that the supremum is achieved at some  $y \in P$ . Duality then implies that  $x \in \partial u_s(y) = \partial u_s^{\star\star}(y)$  ([HL2], Theorem 1.4.1, p. 47), and essential smoothness (Lemma 6.1) implies that  $y \in P \setminus \partial P$ . Hence by Lemma 4.1 we have  $\partial u_s(P \setminus \partial P) = \nabla u_s^{\star\star}(P \setminus \partial P) = \mathbb{R}^n$ , as desired.

Our goal in this section is to prove the surjectivity of  $\nabla u_s^{\star\star}$  by using an alternative homotopy argument. An advantage of this approach is that in the course of the

proof we also obtain that  $u_s^{**}$  and  $\nabla u_s^{**}$  are continuous in  $s$ . This fact is useful when studying the structure of the singular locus of  $\psi$  (see the discussion in §1 just before §§1.1). We believe that albeit being more involved, this approach has its own interest, and might also find applications in situation where the families of functions studied are less explicit.

First, we state an elementary lemma that describes the gradient image of  $u_s$ .

LEMMA 7.1. *For each  $s \in \mathbb{R}_+$ , we have*

$$\nabla u_s(P \setminus \partial P) = \mathbb{R}^n.$$

*Proof.* Recall that by (20) we have

$$u_0 - u_G \in C^\infty(P). \quad (39)$$

From the explicit formula (19) for  $u_G$  and the Delzant conditions one may verify directly that  $\nabla u_G(P \setminus \partial P) = \mathbb{R}^n$ . Note that (19) and (39) also imply that  $\nabla u_s$  is, as a smooth map of  $P \setminus \partial P$  into  $\mathbb{R}^n$ , properly homotopic to  $\nabla u_0$ , that is itself properly homotopic to  $\nabla u_G$ . It follows that the topological degree of  $\nabla u_s$  equals that of  $\nabla u_G$ . Since  $\nabla u_G$  is a bijection we have  $\deg(\nabla u_s) = 1$ . It follows that  $\nabla u_s : P \setminus \partial P \rightarrow \mathbb{R}^n$  is surjective (see, e.g., [OR], Chapter 3, or [G2], Theorems 3.6.6, 3.6.8).  $\square$

LEMMA 7.2. *Let  $s > 0$ .*

(i) *We have,*

$$\nabla u_s^{**}(P \setminus \partial P) = \partial u_s(P \setminus \partial P) = \mathbb{R}^n.$$

(ii) *Moreover,*

$$\partial u_s(P \setminus (\partial P \cup A_s)) = \mathbb{R}^n, \quad (40)$$

*and*

$$\nabla u_s^{**} = \nabla u_s = \partial u_s, \quad \text{on } P \setminus (\partial P \cup A_s).$$

*Proof.* (i) Equation (34) implies that  $\partial u_s^{**}(P \setminus \partial P) \subset \partial u_s(P \setminus \partial P)$ . On the other hand, if  $\partial u_s(y) \neq \emptyset$  then by Lemma 5.1 and its proof we have  $u_s^{**}(y) = u_s(y)$  and  $\partial u_s^{**}(y) = \partial u_s(y)$ , hence  $\partial u_s(P \setminus \partial P) \subset \partial u_s^{**}(P \setminus \partial P)$ . Thus,

$$\partial u_s(P \setminus \partial P) = \partial u_s^{**}(P \setminus \partial P). \quad (41)$$

Since (by Lemma 4.1)  $\nabla u_s^{**}(P \setminus \partial P) = \partial u_s^{**}(P \setminus \partial P)$ , it suffices to show

$$\nabla u_s^{**}(P \setminus \partial P) = \mathbb{R}^n. \quad (42)$$

To that end, we will again rely on degree theory, however now for continuous maps of  $S^n$ . We extend  $\nabla u_s^{**}$  to a map  $G(s, y) : S^n \rightarrow S^n$ , where  $S^n = \mathbb{R}^n \cup \{\infty\}$ , defined by

$$G(s, y) = \begin{cases} \nabla u_s^{**}(y), & y \in P, \\ \infty, & y \in S^n \setminus P. \end{cases}$$

Observe that by Lemma 6.1 for each fixed  $s$  the map  $G(s, \cdot) : S^n \rightarrow S^n$  is continuous, and

$$G(s, y) = \begin{cases} \nabla u_s^{**}(y), & y \in P \setminus \partial P, \\ \infty, & y \in \overline{S^n \setminus P}. \end{cases} \quad (43)$$

Next, we prove that this map is also continuous in  $s$ .

**CLAIM 7.3.** *As continuous maps,  $\nabla u_s^{**}$  is homotopic to  $\nabla u_0$ .*

*Proof.* By definition, we need to show that the map  $G : \mathbb{R}_+ \times S^n \rightarrow S^n$  defined above is continuous. As pointed out, it remains to show that for each fixed  $y \in S^n$ , the map  $G(\cdot, y) : \mathbb{R}_+ \rightarrow S^n$  is continuous. Observe that by (43) it only remains to treat the case  $y \in P \setminus \partial P$ .

Fix  $y \in P \setminus \partial P$  as well as  $s > 0$ . Let  $\{s_j\}_{j \geq 1} \subset \mathbb{R}_+$  satisfy  $\lim_{j \rightarrow \infty} s_j = s$ . By convexity and Lemma 4.1 (ii) we have for each  $j \geq 1$ ,

$$u_{s_j}^{**}(y') \geq u_{s_j}^{**}(y) + \langle y' - y, \nabla u_{s_j}^{**}(y) \rangle, \quad \forall y' \in P. \quad (44)$$

Let  $y^*$  be any limit point of  $\{\nabla u_{s_j}^{**}(y)\}_{j \geq 1}$ . If for each  $v \in P$  the map  $s \mapsto u_s^{**}(v)$  were continuous then taking the limit as  $j$  tends to infinity in (44) would imply

$$u_s^{**}(y') \geq u_s^{**}(y) + \langle y' - y, y^* \rangle, \quad \forall y' \in P.$$

implying that  $y^* \in \partial u_s^{**}(y)$ . Lemma 4.1 would then imply that  $y^* = \nabla u_s^{**}(y)$ , proving the Claim. Now, to prove the continuity of  $u_s^{**}(v)$  in  $s$  it suffices to use the representation formula (29). On the one hand,

$$\begin{aligned} u_{s'}^{**}(y) &= \inf \left\{ \sum_{i=1}^{n+1} \lambda_i u_{s'}(y_i) : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\} \\ &= \inf \left\{ \sum_{i=1}^{n+1} \lambda_i u_s(y_i) + (s' - s) \sum_{i=1}^{n+1} \lambda_i \dot{u}_0(y_i) : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\} \\ &\geq \inf \left\{ \sum_{i=1}^{n+1} \lambda_i u_s(y_i) : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\} \\ &\quad + \inf \left\{ (s' - s) \sum_{i=1}^{n+1} \lambda_i \dot{u}_0(y_i) : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\} \\ &\geq u_s^{**}(y) + (s' - s) \min_P \dot{u}_0. \end{aligned}$$

And on the other hand,

$$\begin{aligned} u_{s'}^{**}(y) &\leq \inf \left\{ \sum_{i=1}^{n+1} \lambda_i u_s(y_i) + (s' - s) \max_P \dot{u}_0 : \lambda \in \Delta_{n+1}, y_i \in P, \sum_{i=1}^{n+1} \lambda_i y_i = y \right\} \\ &= u_s^{**}(y) + (s' - s) \max_P \dot{u}_0. \end{aligned}$$

Hence,

$$|u_{s'}^{**}(v) - u_s^{**}(v)| < (s' - s) \cdot \|\dot{u}_0\|_{C^0(P)} < C(s' - s),$$

as desired.  $\square$

Thus, applying Lemma 7.1 for  $s < T_{\text{span}}^{\text{cvx}}$ , it follows that  $\deg \nabla u_s^{**} = 1$ , and hence that  $\nabla u_s^{**}$  is surjective (see, e.g., [H], §2.2). This implies (42).

(ii) The second part of the statement is already contained in Lemma 5.1 and its proof (see (37)). The first part follows from (i) and Lemma 5.1.  $\square$

## 8. MONOTONICITY AND CONTINUITY OF $A_s$

The following two lemmas establish the continuity of the set-valued map  $s \mapsto A_s$ , its strict monotonicity, and identify its asymptotic limit  $A_\infty$ . Later, in Lemma 13.3, the first two facts are used to establish a strict lower bound on the Monge-Ampère mass of the Legendre transform potential, while the third fact is used to show an a priori upper bound.

**LEMMA 8.1.** *Let  $s_2 > s_1 > 0$ . Then*

$$\overline{A_{s_1}} \setminus \partial P \subset \text{int } A_{s_2}. \quad (45)$$

*In addition,*

$$A_\infty := \bigcup_{s>0} A_s = \{y \in P : \dot{u}_0(y) \neq (\dot{u}_0)^{**}(y)\}, \quad (46)$$

*Proof.* We claim that whenever  $s_1 < s_2$  one has

$$A_{s_1} \subset A_{s_2}. \quad (47)$$

Let  $y \in A_{s_1} \setminus \partial P$ . By Lemma 5.1 then  $\partial u_{s_1}(y) = \emptyset$ . Hence,  $\nabla u_{s_1}(y)$  is not a subgradient for  $u_{s_1}$  at  $y$ , so there exists  $y' \in P \setminus \{y\}$  such that

$$u_{s_1}(y') < \langle y' - y, \nabla u_{s_1}(y) \rangle + u_{s_1}(y). \quad (48)$$

On the other hand, since  $u_0$  is strictly convex,

$$u_0(y') > \langle y' - y, \nabla u_0(y) \rangle + u_0(y), \quad (49)$$

and it follows that

$$\dot{u}_0(y') < \langle y' - y, \nabla \dot{u}_0(y) \rangle + \dot{u}_0(y). \quad (50)$$

Multiplying (50) by  $s_2 - s_1 > 0$  and adding to (48) then implies that  $\partial u_{s_2}(y) = \emptyset$ , proving (47).

To prove (45) it remains to show that  $\partial A_{s_1} \subset \text{int } A_{s_2}$ . Assume that  $y \in \partial A_{s_1}$ . Then there exists some  $y' \in P \setminus \{y\}$  such that

$$u_{s_1}(y') = \langle y' - y, \nabla u_{s_1}(y) \rangle + u_{s_1}(y). \quad (51)$$

It then follows from (49) that (50) holds. As before, this implies that

$$u_{s_2}(y') < \langle y' - y, \nabla u_{s_2}(y) \rangle + u_{s_2}(y),$$

hence  $y \in \text{int } A_{s_2}$ , as desired.

Next, note that (50) implies that  $\partial \dot{u}_0(y) = \emptyset$ , hence

$$A_s \subset \{y \in P : \dot{u}_0(y) \neq (\dot{u}_0)^{**}(y)\},$$

for every  $s > 0$ .

Conversely, given  $y \in P$  such that  $\partial \dot{u}_0(y) = \emptyset$ , let  $y' \in P \setminus \partial P$  be such that (50) holds. It follows that (48) must hold for  $s_1 > 0$  large enough, i.e.,  $\partial u_{s_1}(y) = \emptyset$  and  $y \in A_{s_1}$ . By (45) then  $y \in A_s$  for every  $s > s_1$  and (46) follows.  $\square$

**LEMMA 8.2.** *The map  $s \mapsto A_s$  is continuous as a set-valued map.*

*Proof.* We will show that the map is both lower and upper semi-continuous. Namely, for given  $s$  and  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $s' \in (s - \delta, s + \delta)$  holds

$$A_s \subset A_{s'} + B(0, \epsilon), \quad (52)$$

and

$$A_{s'} \subset A_s + B(0, \epsilon), \quad (53)$$

where  $B(0, \epsilon) := \{y \in \mathbb{R}^n : |y| < \epsilon\}$ , and the sum is in the sense of Minkowski.

First, we prove the lower semi-continuity. Let  $y \in A_s$ . We start with the special case where  $\delta_s := d(A_s, \partial P) > 0$ , where  $d$  denotes the Euclidean distance function. Hence, by the preceding Lemma we may assume that  $s' < s$ . Consider the function  $F : \mathbb{R}_+ \times (P \setminus \partial P) \times P \rightarrow \mathbb{R}$  defined by

$$F(\sigma, v, w) := u_\sigma(w) - u_\sigma(v) - \langle w - v, \nabla u_\sigma(v) \rangle.$$

Note that  $F$  is smooth on its domain. Let  $G : \mathbb{R}_+ \times P \setminus \partial P \rightarrow \mathbb{R}$  be defined by

$$G(\sigma, v) := \min_{w \in P} F(\sigma, v, w).$$

Then  $G$  is continuous. Since  $F$  is uniformly Lipschitz on  $[s-1, s] \times P \setminus B(\partial P, \delta_s) \times P$ , it follows that  $G$  is uniformly continuous on  $[s-1, s] \times P \setminus B(\partial P, \delta_s)$  (we assume without loss of generality that  $s > 1$ ), where  $B(\partial P, \delta_s)$  denotes a  $\delta_s$ -neighborhood of  $\partial P$  in  $P$ . Note that in general  $A_\sigma \setminus \partial P = \{y \in P \setminus \partial P : G(\sigma, y) < 0\}$ . By the preceding Lemma we have  $d(A_{s'}, \partial P) \geq \delta_s$ . Hence, under our assumption that  $\delta_s > 0$ , it follows that there exists some  $\delta > 0$  such that (52) holds whenever  $|s' - s| < \delta$ .

We now turn to the general case, and assume  $A_s \cap \partial P \neq \emptyset$ . By the previous case, we already know that we may choose  $\delta = \delta(\delta_s) > 0$  such that for every  $y \in A_s$  with  $d(y, \partial P) \geq \delta_s$  there exists some  $y' \in A_{s'}$  satisfying  $|y - y'| < \epsilon$ . However, we need to show that  $\delta$  does not tend to zero as  $\delta_s$  does.

To that end, let us assume that  $y$  is close to the boundary of  $P$ , but not in  $\partial P$ . We will return to the case  $y \in \partial P$  later. We assume also, without loss of generality, that  $l_1(y) = \min_{i \in \{1, \dots, d\}} l_i(y)$ , with  $l_1(y) < l_j(y)$  for all  $j \in \{2, \dots, d\}$ . We complement  $l_1$  with  $n-1$  other functions, which for simplicity of notation we assume are  $l_2, \dots, l_n$ , in such a manner that  $l_1, \dots, l_n$  form a coordinate system in  $\mathbb{R}^n$ .

Now, we consider the function  $F$  with its last argument restricted to the line  $L := \{v \in P : l_j(v) = l_j(y), j = 2, \dots, n\}$  in  $P$  that passes through  $y$  and is perpendicular to the face

$$\mathcal{F} := \{v : l_1(v) = \langle v, v_1 \rangle - \lambda_1 = 0\},$$

i.e.,

$$H(\sigma, y, t) := u_\sigma(y + tv_1) - u_\sigma(y) - \langle tv_1, \nabla u_\sigma(y) \rangle, \quad t \in [-C_1, C_2],$$

with  $C_1 = C_1(y) > 0$  proportional to  $d(y, \partial P)$  (up to some uniform constant), and such that  $l_1(y - C_1 v_1) = 0$ , and with  $C_2 = C_2(y) > 0$  uniformly bounded from above and away from zero, (under the assumption that  $l_1(y) < \delta_s$ ), and such that  $y + C_2 v_1 \in \partial P$ .

The last term in  $H$ , computed with respect to the coordinates  $l_1, \dots, l_n$ , equals

$$\langle w - v, \nabla u_\sigma(v) \rangle = a_\sigma(t) \log l_1(y) + b_\sigma(t), \quad (\text{where } w = v + t v_1),$$

for some uniformly bounded functions  $a_\sigma(t), b_\sigma(t)$  of  $t \in [-C_1, C_2]$ , and with  $a_\sigma(t) < 0$  for  $t > 0$ , and  $a_\sigma(t) \geq 0$  for  $t \leq 0$ . Hence,

$$H(\sigma, y, t) < C' - a_\sigma(t) \log l_1(y),$$

for some  $C' = C'(\sigma) > 0$ . It then follows that if  $y$  is taken close enough to  $\mathcal{F}$ , i.e.,  $-\log l_1(y)$  is large enough, then for some  $t \in [0, C_2]$  we will have  $H(\sigma, y, t) < 0$ .

Note that  $C' = C'(\sigma)$  is uniformly bounded for  $\sigma$  in some small neighborhood in  $\mathbb{R}_+$  (independently of  $y$ ). Hence, the above arguments imply that given  $\epsilon > 0$ , we may find  $\delta > 0$  such that whenever  $|s' - s| < \delta$  we may also find  $y' \in P \setminus \partial P$  with  $|y' - y| < \epsilon$  and such that  $H(s', y', t') < 0$  with  $t' \in (0, C_2(y'))$ . In particular, we found a  $w' := y' + t' v_1$  satisfying  $F(s', y', w') < 0$ , hence  $y' \in A_{s'}$ , as desired.

Finally, if  $y \in \partial P$ , since  $A_s$  is open in the relative topology of  $P$  we may choose  $\tilde{y} \in A_s \cap P \setminus \partial P$  with  $|\tilde{y} - y| < \epsilon/2$ . We may then carry out the arguments above for  $\tilde{y}$  to find  $\delta > 0$  such that whenever  $|s' - s| < \delta$  there exists  $y' \in A_{s'}$  with  $|y' - \tilde{y}| < \epsilon/2$ . Hence, once again,  $y \in A_{s'} + B(0, \epsilon)$ , and this concludes the proof of the lower semi-continuity.

The proof of the upper semi-continuity (53) involves similar arguments, by switching the roles of  $s$  and  $s'$ .  $\square$

## 9. ON THE INVERTIBILITY OF $\nabla u_s^{**}$ AND STRICT CONVEXITY OF $u_s^{**}$

In this section we describe the set on which  $u_s^{**}$  is strictly convex.

**LEMMA 9.1.**  *$\nabla u_s^{**}$  is invertible on  $\text{int}(P \setminus (\partial P \cup A_s))$ . Moreover, if  $y \in \text{int}(P \setminus (\partial P \cup A_s))$  then*

$$u_s(y') > \langle y' - y, \nabla u_s(y) \rangle + u_s(y), \quad \forall y' \in P \setminus \{y\}, \quad (54)$$

and

$$u_s^{**}(y') > \langle y' - y, \nabla u_s^{**}(y) \rangle + u_s^{**}(y), \quad \forall y' \in P \setminus \{y\}. \quad (55)$$

*Proof.* Note that (55) implies (54), since  $u_s^{**}(y') \leq u_s(y')$ ,  $u_s^{**}(y) = u_s(y)$ , and  $\nabla u_s^{**}(y) = \nabla u_s(y)$ . In addition, the invertibility statement also follows from (55).

Let  $y \in \text{int}(P \setminus (\partial P \cup A_s))$ . By convexity of  $u_s^{**}$ ,

$$u_s^{**}(y') \geq \langle y' - y, \nabla u_s^{**}(y) \rangle + u_s^{**}(y), \quad \forall y' \in P. \quad (56)$$

Suppose now that equality holds in (56) for some  $y' \in P$  (by Lemma 6.1 necessarily  $y' \in P \setminus \partial P$ ). By convexity and differentiability of  $u_s^{**}$  then

$$\nabla u_s^{**}(y') = \nabla u_s^{**}(y) = \nabla u_s(y). \quad (57)$$

Since  $\partial u_s(y) \neq \emptyset$  then  $u_s$  is convex at  $y$ , i.e.,  $\nabla^2 u_s(y) \geq 0$ . We claim that in fact

$$\nabla^2 u_s(y) > 0. \quad (58)$$

Before proving (58) let us prove that it implies that  $y' = y$ , and hence proves (55). Indeed, consider the function

$$F(t) := u_s^{**}(ty + (1-t)y').$$

This function is convex for  $t \in [0, 1]$ , and strictly convex for  $t \in (1-\epsilon, 1]$  for some  $\epsilon \in (0, 1)$  (here we use (58) and that  $F(t) = u_s(ty + (1-t)y')$  for  $t$  near 1 since then  $ty + (1-t)y' \in \text{int}(P \setminus (\partial P \cup A_s))$ ). Thus,

$$\langle \nabla u_s^{**}(y'), y - y' \rangle = F'(0) < F'(1) = \langle \nabla u_s^{**}(y), y - y' \rangle,$$

By (57) we must have then  $y' = y$ .

We return to proving (58). Assume on the contrary that  $\langle \nabla^2 u_s(y) \xi, \xi \rangle = 0$  for some  $\xi \in \mathbb{R}^n \setminus \{0\}$ . First, observe that since  $\nabla^2 u_0 > 0$ , it follows that for every  $\epsilon > 0$  one has  $\langle \nabla^2 u_{s+\epsilon}(y) \xi, \xi \rangle < 0$ . Hence,  $\partial u_{s+\epsilon}(y) = \emptyset$ , i.e.,  $y \in A_{s+\epsilon} \setminus \partial P$ . By Lemma 8.1 it follows that  $y \in \overline{A_s} \setminus \partial P$ , contradicting our assumption.  $\square$

A certain converse of the preceding Lemma is given by the following.

**LEMMA 9.2.** (i) Let  $y \in P \setminus \partial P$ . Assume that  $u_s^{**}$  is strictly convex in some neighborhood of  $y$ . Then  $y \in \text{int}(P \setminus (\partial P \cup A_s))$ .

(ii) Let  $y \in A_s \setminus \partial P$ . Then there exists a line in  $\overline{A_s} \setminus \partial P$  passing through  $y$  and intersecting  $\partial A_s \setminus \partial P$ , along which  $\nabla u_s^{**}$  is constant.

*Proof.* We only prove (ii) since it implies (i). However, note that (i) is also a consequence of Lemma 7.2 (ii) that implies that  $\partial u_s^{**}(A_s \setminus \partial P) = \emptyset$ .

Let  $y \in A_s \setminus \partial P$ . Consider the tangent hyperplane at  $y$ , given by the equation  $l(y') = u_s^{**}(y) + \langle \nabla u_s^{**}(y), y' - y \rangle$ . By convexity  $l \leq u_s^{**}$  on  $P$ . If one has  $l < u_s^{**} = u_s$  on  $P \setminus A_s$ , then by compactness for some  $\epsilon > 0$  one has also  $l + \epsilon < u_s^{**}$  there. However, by (27) a fortiori  $u_s^{**}$  equals the supremum of all affine functions majorized by  $u_s$  over  $P \setminus A_s$ . But then one would obtain a contradiction to  $l(y) = u_s^{**}(y)$ . It follows that for some  $y' \in P \setminus A_s$  one has  $l(y') = u_s^{**}(y')$ . Note that by the essential smoothness of  $u_s^{**}$  proved in Lemma 6.1 we must have  $y' \in P \setminus \partial P$ . Since  $l$  is affine, convexity then implies that for each point on the line segment connecting  $y$  to  $y'$  one has  $l = u_s^{**}$ . It follows that  $l$  is the tangent hyperplane to  $u_s^{**}$  at each one of those points. Since  $y' \notin \text{int } A_s$  it follows that the line connecting  $y$  and  $y'$  intersects  $\partial A_s \setminus \partial P$ , proving our claim.

As an alternative proof, one may also show that  $u_s^{**}$  is affine on the polyhedron  $\text{co}\{y_i : i \in I\}$  containing  $y$  and generated by the points called upon by  $y$  ([HL2], Theorem 1.5.5, p. 52).  $\square$

Note that the preceding Lemma does not quite give a foliation of  $A_s \setminus \partial P$  by lines along which  $\nabla u_s^{**}$  is constant. The rank of  $\ker \nabla^2 u_s^{**}$  may jump in  $A_s \setminus \partial P$  and so there may be more than one line with that property passing through a given point. Instead,  $A_s \setminus \partial P$  is partitioned into maximal sets along which  $\nabla u_s^{**}$  is constant (see (76), (91) and §13).

Yet, as a corollary of Lemma 9.2 (ii) (or of Lemma 7.2 (ii)) we have

$$\partial u_s^{**}(A_s \setminus \partial P) \subset \partial u_s^{**}(\partial A_s \setminus \partial P)$$

(cf. [HN], Theorem I). Together with Theorem 2.4 it follows that over  $A_s \setminus \partial P$  the function  $u_s^{**}$  is the solution of the HRMA (in dimension  $n$ )  $\text{MA } u_s^{**} = 0$ . The difficulty in replacing the regularity results of §4–6 by the general  $C^{1,1}$  regularity results for the Dirichlet problem for the HRMA is that, aside from the fact that  $A_s$  may be disconnected and  $P \setminus A_s$  might not be convex, it is not clear that  $\partial A_s$  will be regular enough to apply the results of [TU, CNS]. Further, its boundary may intersect  $\partial P$ , in which case we would not be able to prescribe the Dirichlet data.

## 10. PARTIAL $C^1$ REGULARITY OF THE LEGENDRE TRANSFORM POTENTIAL

Recall the definition of the Legendre transform potential,

$$\psi(s, x) = \psi_s(x) := u_s^*(x), \quad s \geq 0, \quad x \in \mathbb{R}^n.$$

It is a one-parameter family of convex functions (in  $x$ ), and for each  $s$  the function  $\psi_s$  is defined and finite on all of  $\mathbb{R}^n$  (see §7). Moreover, it is a convex function of  $(s, x)$ : by definition

$$\psi(s, x) = \sup_{y \in P} [\langle x, y \rangle - u_0(y) - s \dot{u}_0(y)] = \sup_{y \in P} [\langle (s, x), (-\dot{u}_0(y), y) \rangle - u_0(y)], \quad (59)$$

i.e.,  $\psi$  is the supremum of linear functions in  $(s, x)$ , hence convex. Observe also that if we would have taken the Legendre transform of  $u_0 + s \dot{u}_0$  with respect to all  $n+1$  variables the resulting function would be equal a.e. to  $+\infty$ .

Set

$$\Sigma_{\text{reg}}(T) := \bigcup_{s \in [0, T]} \{s\} \times \partial u_s(\text{int}(P \setminus (\partial P \cup A_s))) \subseteq [0, T] \times \mathbb{R}^n.$$

Note that by Lemma 4.1 we could have replaced the subdifferential with the gradient in the definition of  $\Sigma_{\text{reg}}(T)$ .

Given a convex function  $f$ , recall that

$$\Delta(f) \tag{60}$$

denotes the set on which  $f$  is finite and differentiable.

The following result states that  $\Sigma_{\text{reg}}(T)$  coincides with the regular locus of  $\psi$ . Moreover, it shows the following partial  $C^1$  regularity for  $\psi$ —the regular set of  $\psi$  is simply the (indexed) union of the regular sets of  $\psi_s$ .

**PROPOSITION 10.1.** *We have*

$$\Sigma_{\text{reg}}(T) = \Delta_T(\psi) := \Delta(\psi) \cap [0, T] \times \mathbb{R}^n.$$

Further,

$$\Delta(\psi) = \bigcup_{s > T_{\text{span}}^{\text{cvx}}} \{s\} \times \Delta(\psi_s).$$

Recall the definition of the singular locus of  $\psi$  (§1)

$$\Sigma_{\text{sing}}(T) := [0, T] \times \mathbb{R}^n \setminus (\Delta(\psi) \cap [0, T] \times \mathbb{R}^n).$$

The proposition gives the following explicit description

$$\Sigma_{\text{sing}}(T) = \bigcup_{s \in [0, T]} \{s\} \times \nabla u_s^{\star\star}(A_s \setminus \partial P) \subseteq [0, T] \times \mathbb{R}^n.$$

Note that  $\Delta(\psi)$  is not in general open (for instance, consider the situation when a sequence of singular points  $\{(s_k, x_k)\}_{k \geq 1} \subset \Sigma_{\text{sing}}(T)$  with  $\lim_{k \rightarrow \infty} s_k = T_{\text{span}}^{\text{cvx}}$  converges to a point in  $\Delta(\psi)$ ). However, it is dense, and its complement has Lebesgue measure zero ([Ro], Theorem 25.5; [RT], Proposition 2.4).

Recall the following duality between differentiability and strict convexity.

**LEMMA 10.2.** (See [Ro], Theorem 26.3.) *A closed proper convex function is essentially strictly convex if and only if its Legendre dual is essentially smooth.*

The proof of the Proposition will be a consequence of two lemmas proved below.

First, we show that  $\psi$  is differentiable in  $x$  on  $\Sigma_{\text{reg}}(T)$ .

**LEMMA 10.3.**  $(s, x) \in \Sigma_{\text{reg}}(T)$  if and only if  $x \in \Delta(\psi_s)$ .

*Proof.* By definition  $\psi_s(x) = \sup_{y' \in P} (\langle x, y' \rangle - u_s^{\star\star}(y'))$  and the supremum is achieved when  $x \in \partial \psi_s^{\star}(y') = \partial u_s^{\star\star}(y') = \nabla u_s^{\star\star}(y')$ . Assume  $(s, x) \in \Sigma_{\text{reg}}(T)$ . We claim that then  $y' \in \text{int}(P \setminus (\partial P \cup A_s))$ . Indeed, by our assumption we know that there exists some  $y \in \text{int}(P \setminus (\partial P \cup A_s))$  such that  $x = \nabla u_s(y)$ . Equation (55) and Lemma 4.1 (ii) then imply that  $\nabla u_s^{\star\star}(y') \neq \nabla u_s^{\star\star}(y)$ , unless  $y' = y$ , i.e.,  $y' = (\nabla u_s^{\star\star})^{-1}(x)$ , as desired. It follows that

$$\psi_s(x) = \langle x, (\nabla u_s^{\star\star})^{-1}(x) \rangle - u_s^{\star\star} \circ (\nabla u_s^{\star\star})^{-1}(x), \quad x \in \Sigma_{\text{reg}}(T). \quad (61)$$

Since  $\partial u_s(\text{int}(P \setminus (\partial P \cup A_s))) = \nabla u_s(\text{int}(P \setminus (\partial P \cup A_s)))$  is an open set, equation (61) holds in a neighborhood of  $x$  (for  $s$  fixed). Since  $u_s^{\star\star}$  equals  $u_s$  on the open set  $\text{int}(P \setminus (\partial P \cup A_s))$ , it is smooth there. Hence, we may differentiate (61) in  $x$  to obtain (see (17))

$$\nabla \psi_s(x) = (\nabla u_s^{\star\star})^{-1}(x), \quad (62)$$

and this is a singleton.

Conversely, assume  $x \in \Delta(\psi_s)$ . Then by Lemma 10.2 it follows that  $\psi_s^{\star}$  is strictly convex in a neighborhood of  $\nabla \psi_s(x)$ . We have  $\nabla \psi_s(x) \in \text{int}(P \setminus (\partial P \cup A_s))$  by Lemma 9.2, and  $x \in \partial u_s^{\star\star}(\text{int}(P \setminus (\partial P \cup A_s))) = \partial u_s(\text{int}(P \setminus (\partial P \cup A_s)))$  by duality (and Lemma 4.1), i.e.,  $(s, x) \in \Sigma_{\text{reg}}(T)$ .  $\square$

Next, we show that wherever  $\psi$  is differentiable in  $x$  it is also differentiable in  $s$ .

**LEMMA 10.4.** *Assume that  $x \in \Delta(\psi_s)$ . Then  $(s, x) \in \Delta(\psi)$ .*

*Proof.* It suffices to show that (61) holds in a neighborhood of  $(s, x)$ . Then the usual derivation of the first variation formula for the Legendre transform (see, e.g., [R], p. 85) gives

$$\frac{\partial \psi}{\partial s}(s, x) = -\frac{\partial u}{\partial s}(s, \nabla_x \psi(s, x)) = -\dot{u}_0 \circ \nabla_x \psi(s, x), \quad (63)$$

implying  $(s, x) \in \Delta(\psi)$ .

By the proof of Lemma 10.3 we already know that (61) holds in a neighborhood of  $x$ , with  $s$  fixed, and that, further, it suffices to show that for some  $\epsilon > 0$  with  $s + \epsilon < T$ ,

$$x \in \nabla u_{s'}(\text{int}(P \setminus (\partial P \cup A_{s'}))), \quad \forall s' \in (s - \epsilon, s + \epsilon), \quad (64)$$

i.e.,  $(s', x) \in \Sigma_{\text{reg}}(T)$ .

First, we claim that if  $y \in \text{int}(P \setminus (\partial P \cup A_s))$  and  $\epsilon > 0$  is sufficiently small then  $y \in \text{int}(P \setminus (\partial P \cup A_{s'}))$  for all  $s' \in (s - \epsilon, s + \epsilon)$ . For that, it suffices to show that for  $\epsilon > 0$  sufficiently small and  $s' \in (s - \epsilon, s + \epsilon)$  there exists a neighborhood  $U_{s,y}$  of  $y$  such that  $\partial u_{s'}(y') \neq \emptyset$  for all  $y' \in U_{s,y}$ . Indeed, by (54),

$$u_s(\tilde{y}) > \langle \tilde{y} - y, \nabla u_s(y) \rangle + u_s(y), \quad \forall \tilde{y} \in P \setminus \{y\}, \quad (65)$$

and moreover the same holds for  $y$  replaced by a sufficiently nearby point. By smoothness of  $\dot{u}_0$  and compactness of  $P$  we also have that

$$\max_{\tilde{y} \in P} |\dot{u}_0(\tilde{y}) - \dot{u}_0(y) - \langle \tilde{y} - y, \nabla \dot{u}_0(y) \rangle| < C,$$

with  $C > 0$  a uniform constant independent of  $y$ . Hence (65) holds with  $s$  replaced by  $s' \in (s - \epsilon, s + \epsilon)$  and with  $y$  replaced by  $y'$  with  $|y - y'| < \epsilon$  for  $\epsilon$  sufficiently small. Thus,  $\partial u_{s'}(y') \neq \emptyset$ , proving our claim.

Now, (64) follows since  $x \in \nabla u_s(\text{int}(P \setminus (\partial P \cup A_s)))$  and the mapping  $s \mapsto \nabla u_0 + s \nabla \dot{u}_0$  is continuous in  $s$ .  $\square$

Combining the results obtained so far we may give a proof of Proposition 1.

*Proof of Proposition 1.* According to Lemma 4.1 for each  $s > 0$  the function  $u_s^{**}$  is finite, and according to Lemma 7.2 its gradient surjects to  $\mathbb{R}^n$ . It follows that its dual  $\psi_s$  must be finite, since for every  $x \in \mathbb{R}^n$  the supremum in the definition of the Legendre transform is necessarily achieved for  $y \in \partial \psi_s(x)$  and any  $y \in P$  satisfying  $\partial u_s^{**}(y) = x$  will do. Convexity then implies that  $\psi_s$  will be Lipschitz continuous (see [RT], Proposition 2.4).

Lemma 10.2, combined with Lemma 6.1, Lemma 9.2, and the fact that  $A_s \neq \emptyset$  for  $s > T_{\text{span}}^{\text{cvx}}$ , implies simultaneously that  $\psi_s$  is essentially strictly convex, and that it is not differentiable. In fact, since  $\psi_s$  is convex and continuous on  $\mathbb{R}^n$  it follows that  $\text{dom}(\partial \psi_s) = \mathbb{R}^n$ , thus  $\psi_s$  is strictly convex. In addition, the exact description of the regular locus of  $\psi_s$  in Lemma 10.3 and the differentiability of  $u_s^{**}$  (Lemma 4.1) imply that the singular locus of  $\psi_s$  is  $\nabla u_s^{**}(A_s \setminus \partial P)$ . This concludes the proof of Proposition 1.

## 11. THE LEGENDRE TRANSFORM POTENTIAL ON THE REGULAR LOCUS

The following Proposition shows that  $\psi$  solves the HRMA wherever it is differentiable. This is part of the statement of Theorem 1.

PROPOSITION 11.1. (i) For  $T < T_{\text{span}}^{\text{cvx}}$ ,

$$\partial\psi([0, T] \times \mathbb{R}^n) = \text{graph of } -\dot{u}_0 \text{ over } P \setminus \partial P. \quad (66)$$

(ii) Let  $T > 0$ . One has

$$\text{MA } \psi|_{\Sigma_{\text{reg}}(T)} = 0. \quad (67)$$

Moreover,

$$\partial\psi(\Sigma_{\text{reg}}(T) \cap \{s\} \times \mathbb{R}^n) = \text{graph of } -\dot{u}_0 \text{ over } \text{int}(P \setminus (\partial P \cup A_s)). \quad (68)$$

(iii)  $\Sigma_{\text{reg}}(T)$  is dense in  $[0, T] \times \mathbb{R}^n$  and its complement has zero Lebesgue measure there.

*Proof.* (i) For  $T < T_{\text{span}}^{\text{cvx}}$  one has using (21),

$$\begin{aligned} \partial\psi(s, x) &= (\dot{\psi}_s(x), \nabla_x \psi_s(x)) = (-\dot{u}_s \circ \nabla_x \psi_s(x), \nabla_x \psi_s(x)) \\ &= (-\dot{u}_0 \circ \nabla_x \psi_s(x), \nabla_x \psi_s(x)). \end{aligned} \quad (69)$$

Equation (66) now follows from  $\nabla_x \psi_s(\mathbb{R}^n) = P \setminus \partial P$ .

(ii) According to Lemmas 10.3 and 10.4 we know that  $\psi$  is differentiable on  $\Sigma_{\text{reg}}(T)$ . The same arguments as in the proof of Lemma 10.3 actually demonstrate that it is smooth on  $\Sigma_{\text{reg}}(T)$ . Differentiating (61) twice then yields (see (18))

$$(\nabla^2 \psi_s(x))^{-1} = \nabla^2 u_s(\nabla \psi_s(x)),$$

showing that  $\psi_s$  is strictly convex at  $x$  whenever  $(s, x) \in \Sigma_{\text{reg}}(T)$ .

Moreover, (61) and the proof of Lemma 10.3 show that  $\psi$  is in fact smooth on  $\Sigma_{\text{reg}}(T)$ . In particular, the second variation formula for the Legendre transform

$$-\ddot{u}_s|_{(\nabla u_s)^{-1}(x)} = \ddot{\psi}_s|_x + \langle \nabla \dot{\psi}_s|_x, \nabla \dot{u}_s|_{(\nabla u_s)^{-1}(x)} \rangle = \left( \ddot{\psi}_s - \frac{1}{2} |\nabla \dot{\psi}_s|_{g_{\psi_s}}^2 \right)(x), \quad (70)$$

holds pointwise for  $x \in \Sigma_{\text{reg}}(T)$ , and there these equations are pointwise equivalent to the HRMA (1) (see, e.g., [R], p. 87; [RZ1], §3). Since  $u_s$  solves the equation on the left hand side,  $\psi$  solves the HRMA on  $\Sigma_{\text{reg}}(T)$ .

To prove (68) observe that (63) holds on  $\Sigma_{\text{reg}}(T)$  (recall the arguments in Lemmas 10.3 and 10.4). Since, by duality,

$$\partial_x \psi(\Sigma_{\text{reg}}(T) \cap \{s\} \times \mathbb{R}^n) = \text{int}(P \setminus (\partial P \cup A_s)),$$

equation (68) follows. Note that (68), together with Theorem 2.4, gives another proof of  $\text{MA } \psi|_{\Sigma_{\text{reg}}(T)} = 0$ , since the graph of  $-\dot{u}_0$  over  $P$  is a set of Lebesgue measure zero in  $\mathbb{R}^{n+1}$ .

(iii) Observe that  $\Sigma_{\text{sing}}(T)$  is a set of Lebesgue measure zero in  $[0, T] \times \mathbb{R}^n$  since by Proposition 10.1 it is precisely the set on which the convex function  $\psi$  is not differentiable.  $\square$

Observe that the proof of Proposition 11.1 (iii) relies on Lemma 7.2 (i) implicitly. Alternatively, one may also use Lemma 7.2 (ii) directly to prove that  $\Sigma_{\text{reg}}(T)$  is dense in  $[0, T] \times \mathbb{R}^n$  without using the characterization of  $\Sigma_{\text{reg}}(T)$  in Proposition 10.1.

## 12. THE SUBDIFFERENTIAL OF THE LEGENDRE TRANSFORM POTENTIAL

In this section we prove upper and lower bounds on the total subdifferential of  $\psi$  (Lemma 12.5) in terms of  $u_s^{**}$  and a partition of  $A_s$ . To that end, we first collect some elementary facts concerning subdifferentials of convex functions of several variables. Then, we give a precise description of the partial  $x$ -subdifferential of  $\psi$  and of the set of reachable partial  $x$  subgradients (Lemma 12.4). This is then combined with the partial  $C^1$  regularity of  $\psi$  to prove Lemma 12.5.

Given a convex function  $f$ , define the set of reachable subgradients by

$$\gamma f(x) := \{x^* : \text{exists } \{x_k\}_{k \geq 1} \subset \Delta(f) \text{ with } \lim_{k \rightarrow \infty} (x_k, \nabla f(x_k)) = (x, x^*)\}. \quad (71)$$

(recall (60)).

**LEMMA 12.1.** (See [HL1], Theorem 6.3.1, p. 285.) *Let  $f$  be a closed proper convex function. Then  $\partial f(x) = \text{co } \gamma f(x)$  for  $x \in \text{dom}(\partial f)$ .*

When  $f(x_1, x_2)$  is convex in both of its arguments  $x_1 \in \mathbb{R}^{m_1}, x_2 \in \mathbb{R}^{m_2}$ , we will denote the reachable partial  $x_1$  subgradients by

$$\gamma_{x_1} f(x_1, x_2) := \gamma f_{x_2}(x_1),$$

where  $f_{x_2}(x_1) := f(x_1, x_2)$  is considered as a function of  $x_1$ . Similarly we also define  $\gamma_{x_2} f(x_1, x_2)$ . We will denote by  $\partial_{x_1} f$  the partial  $x_1$  subdifferential of  $f$  considered as a function of  $x_1$  alone,

$$\partial_{x_1} f(x_1, x_2) := \partial f_{x_2}(x_1).$$

Similarly we also define  $\partial_{x_2} f$ .

We will need the following elementary consequence of Lemma 12.1.

**LEMMA 12.2.** *Let  $f(x_1, x_2)$ ,  $x_1 \in \mathbb{R}^{m_1}, x_2 \in \mathbb{R}^{m_2}$ , be a closed proper convex function on  $\mathbb{R}^{m_1+m_2}$ . Assume that  $f$  is sub-differentiable, and differentiable in  $x_1$ . Then wherever  $f$  is finite*

$$\partial f = \nabla_{x_1} f \times \partial_{x_2} f. \quad (72)$$

*Proof.* Fix  $x = (x_1, x_2)$ . By Lemma 12.1, we have

$$\begin{aligned} \partial f(x_1, x_2) &\supset \{(x_1^*, x_2^*) : x_1^* \in \gamma_{x_1} f(x_1, x_2), x_2^* \in \gamma_{x_2} f(x_1, x_2)\} \\ &= \nabla_{x_1} f(x) \times \gamma_{x_2} f(x_1, x_2), \end{aligned} \quad (73)$$

Indeed one takes sequences of points where  $f$  is differentiable and for these points  $\nabla f = (\nabla_{x_1} f, \nabla_{x_2} f)$ . This is possible since a closed proper function is differentiable on a dense set in the interior of  $\text{dom}(f)$ , the set where it is sub-differentiable ([Ro], Theorem 25.5). Lemma 12.1 implies that  $\partial f(x)$  is a convex set, and by applying this Lemma yet once more, that is taking convex hulls in (73), we obtain

$$\partial f(x_1, x_2) \supset \nabla_{x_1} f(x) \times \partial_{x_2} f(x).$$

From the definitions one may verify that

$$\partial f(x_1, x_2) \subset \partial_{x_1} f(x) \times \partial_{x_2} f(x) = \nabla_{x_1} f(x) \times \partial_{x_2} f(x), \quad (74)$$

and this completes the proof.  $\square$

The following lemma describes a general property of convex functions of several variables, that we prove using the concept of reachable subgradients. It also appears, with a different proof using the Hahn-Banach theorem, in [A], Proposition 2.4.

**LEMMA 12.3.** *The projection  $\pi_x : \partial\psi(s, x) \mapsto \partial_x\psi(s, x)$  is surjective.*

*Proof.* First, as in (74), we have

$$\partial\psi(s, x) \subset \partial_x\psi(s, x) \times \partial_s\psi(s, x), \quad (75)$$

and so  $\pi_x$  indeed maps  $\partial\psi(s, x)$  into  $\partial_x\psi(s, x)$ . The set  $\partial\psi(s, x) \subset \mathbb{R}^{n+1}$  is convex. Therefore, by Lemma 12.1, it suffices to verify that the projection  $\pi_x$  surjects to  $\gamma_x\psi(s, x)$ .

Let  $y \in \gamma_x\psi(s, x)$ . Then there exists a sequence of points  $\{x_i\}_{i \geq 1}$  converging to  $x$  and such that  $\nabla_x\psi(s, x_i)$  exists and  $\lim_{i \rightarrow \infty} \nabla_x\psi(s, x_i) = y$ . By Lemma 12.2

$$\lim_{i \rightarrow \infty} \partial\psi(s, x_i) = \lim_{i \rightarrow \infty} \partial_s\psi(s, x_i) \times \{\nabla_x\psi(s, x_i)\},$$

from which  $\pi_x(\lim_{i \rightarrow \infty} \partial\psi(s, x_i)) = y$ . By the upper semi-continuity of the subdifferential mapping ([Ro], Corollary 24.5.1) then  $\lim_{i \rightarrow \infty} \partial\psi(s, x_i) \subset \partial\psi(s, x)$ . We conclude that  $y \in \pi_x(\partial\psi(s, x))$ , as claimed.  $\square$

For each  $(s, y) \in \mathbb{R}_+ \times A_s \setminus \partial P$ , set

$$Q(s, y) := \{v \in P : \nabla u_s^{**}(v) = \nabla u_s^{**}(y)\} \subset \overline{A_s} \setminus \partial P \quad (76)$$

(the inclusion is implied by Lemma 6.1 and the results of §9), and

$$Y(s, y) := Q(s, y) \cap (\partial A_s \setminus \partial P) = Q(s, y) \cap \partial A_s. \quad (77)$$

Note that  $Q(s, y)$  is the projection of the intersection of the tangent hyperplane to the graph of  $u_s^{**}$  at  $y$  with the graph itself. Thus, convexity of  $u_s^{**}$  implies that  $Q(s, y)$  is closed and convex, and

$$Q(s, y) = \text{co } Y(s, y) \subset P \quad (78)$$

(here we use that any point in  $\text{int}(A_s \setminus \partial P)$  is a convex combination of called upon points lying in  $\partial A_s$ , see the proof of Lemma 4.1).

Note also that  $\{y_i\}_{i \in I} \subset Y(s, y)$ . Recall that the the set  $\{y_i\}_{i \in I}$  of points called upon by  $y$  obtained in the proof of Lemma 4.1 was not necessarily the unique collection of at most  $n+1$  points satisfying (30) and (32). The set  $Y(s, y)$  will serve as a more ‘canonical’ substitute for  $\{y_i\}_{i \in I}$ .

**LEMMA 12.4.** *Let  $y \in A_s \setminus \partial P$ , let  $Y(s, y)$  be given by (77), and let  $x = \nabla u_s^{**}(y)$ . Then*

$$Y(s, y) = \gamma_x\psi(s, x), \quad (79)$$

and

$$Q(s, y) = \partial_x\psi(s, x). \quad (80)$$

*Proof.* Let  $v \in Y(s, y)$ . Since  $v \in \partial A_s \setminus \partial P$  (Claim 4.2) there exists a sequence  $\{v_k\}_{k \geq 1} \subset \text{int}(P \setminus (\partial P \cup A_s))$  with  $\lim_{k \rightarrow \infty} v_k = v$ . By (62)  $\nabla \psi_s$  exists and is invertible on  $\text{int}(P \setminus (\partial P \cup A_s))$ . So for each  $k$  there exists a unique  $x_k$  with

$$\nabla u_s(v_k) = (\nabla \psi_s)^{-1}(v_k) = x_k. \quad (81)$$

Therefore,  $\lim_{k \rightarrow \infty} x_k = \nabla u_s(v) = x$ , and  $\lim_{k \rightarrow \infty} \nabla \psi_s(x_k) = v$ . It follows that  $Y(s, y) \subset \gamma_x \psi(s, x)$ .

Conversely, if  $v \in \gamma_x \psi(s, x)$  then by definition there exists a sequence  $\{x_k\}_{k \geq 1}$  in  $\Delta(\psi_s)$  with  $\lim_{k \rightarrow \infty} \nabla \psi_s(x_k) = v$ . By Lemma 10.3  $(s, x_k) \in \Sigma_{\text{reg}}(T)$  for all  $k$ . By duality then  $\nabla \psi_s(x_k) \in \text{int}(P \setminus (\partial P \cup A_s))$ . Since  $(s, x) \in \Sigma_{\text{sing}}(T)$  we must have  $\partial_x \psi(s, x) \ni v \notin \text{int}(P \setminus (\partial P \cup A_s))$ . We conclude that  $v \in \partial A_s$ . Finally,  $\partial_x \psi(s, x) \subset Q(s, y)$ : if  $v \in \partial_x \psi(s, x)$  then by duality,  $x \in \partial u_s^*(v)$  and by Lemma 4.1  $x = \nabla u_s^*(v)$ . Therefore,  $v \in Y(s, y)$ , implying (79).

Taking convex hulls in (79) and invoking (78) and Lemma 12.1 implies (80).  $\square$

**LEMMA 12.5.** *Let  $y \in A_s \setminus \partial P$ , and set  $x := \nabla u_s^*(y)$ . Then*

$$\text{co}\{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s, x)\} \subset \partial \psi(s, x) \subset \text{co}\{(-\dot{u}_0(v), v) : v \in \partial_x \psi(s, x)\}. \quad (82)$$

*Proof.* First, we will show that

$$\{(-\dot{u}_0(v), v) : v \in \gamma_x \psi(s, x)\} \subset \gamma \psi(s, x). \quad (83)$$

Together with Lemma 12.1 this implies the first inclusion in (82). For that, let  $v \in Y(s, y)$ . Let  $\{v_k\}_{k \geq 1}$  and  $\{x_k\}_{k \geq 1}$  be as in the proof of Lemma 12.4. By Lemma 10.4 we have  $(s, x_k) \in \Delta(\psi)$  and

$$\nabla \psi(s, x_k) = (-\dot{u}_0 \circ \nabla \psi_s(x_k), \nabla \psi_s(x_k)).$$

Hence, by (81)  $\lim_{k \rightarrow \infty} \nabla \psi(s, x_k) = (-\dot{u}_0(v), v) \in \gamma \psi(s, x)$ , and (83) follows.

Next, let  $\tilde{v} \in \gamma \psi(s, x) \subset \partial \psi(s, x)$ , and let  $\{(s_k, x_k)\}_{k \geq 1} \subset \Delta(\psi)$  satisfy

$$\lim_{k \rightarrow \infty} (s_k, x_k) = (s, x), \quad \lim_{k \rightarrow \infty} \nabla \psi(s_k, x_k) = \tilde{v}.$$

Then, as in (16)-(17) (since we are in  $\Delta(\psi)$ ),

$$\nabla \psi(s_k, x_k) = (\dot{\psi}(s_k, x_k), \nabla_x \psi(s_k, x_k)) = (-\dot{u}_0 \circ \nabla_x \psi_{s_k}(x_k), \nabla_x \psi_{s_k}(x_k)). \quad (84)$$

Let  $v := \lim_{k \rightarrow \infty} \nabla_x \psi(s_k, x_k) \in \partial_x \psi(s, x)$  (making use of (75)). Then (84) implies that

$$\tilde{v} = (-\dot{u}_0(v), v).$$

The second inclusion in (82) now follows from Lemma 12.1.  $\square$

We note that an alternative proof, starting from the formula (59), may be given. Also, the lower bound can in fact be shown to be sharp. We do not go into the details here since we will not need these facts in the present article.

## 13. MONGE-AMPÈRE MEASURE OF THE LEGENDRE TRANSFORM POTENTIAL

We are now in a position to complete the proof of Theorem 1, and show that the Legendre transform potential  $\psi$  is not a weak solution of the HRMA (1) for  $T > T_{\text{span}}^{\text{cvx}}$ .

Theorem 1 is a direct corollary of the following result, stating that the Monge-Ampère measure of  $\psi$  charges the singular locus of  $\psi$  with positive mass.

**PROPOSITION 13.1.** *Let  $T > T_{\text{span}}^{\text{cvx}}$ . Then*

$$\int_{[0,T] \times \mathbb{R}^n} \text{MA } \psi = \int_{\Sigma_{\text{sing}}(T)} \text{MA } \psi > 0. \quad (85)$$

The proof of Proposition 13.1 will occupy most of this section.

Set

$$\tilde{Q}(s, y) := \text{co} \{ (-\dot{u}_0(v), v) : v \in Y(s, y) \} \subset \mathbb{R} \times P. \quad (86)$$

Note that by Lemmas 12.4 and 12.5 this convex set is contained in  $\partial\psi(s, \nabla u_s^{\star\star}(y))$ .

Let

$$q(s, y) := \dim Q(s, y) \leq n.$$

Recall that

$$\{y_i\}_{i \in I} \subset Y(s, y) \subset Q(s, y),$$

and that  $|I| > 1$ , whenever  $y \in A_s$ . Hence, when  $y \in A_s$ , the set  $Q(s, y)$  contains at least a line, and  $q(s, y) \geq 1$ . By (78) we have  $\pi_P(\tilde{Q}(s, y)) = Q(s, y)$ . Hence,  $q(s, y) \leq \dim \tilde{Q}(s, y) \leq q(s, y) + 1$ .

**LEMMA 13.2.** *Let  $y \in A_s \setminus \partial P$ . Then*

$$U(s, y) := \pi_P(\tilde{Q}(s, y) \setminus \{(-\dot{u}_0(v), v) : v \in Q(s, y)\}) \subset Q(s, y) \quad (87)$$

*is a set with a non-empty interior relative to  $Q(s, y)$ . If  $\dim \tilde{Q}(s, y) = q(s, y)$  then*

$$\tilde{Q}(s, y) \cap \{(-\dot{u}_0(v), v) : v \in U(s, y)\} = \emptyset. \quad (88)$$

*Proof.* We may assume that  $\dim \tilde{Q}(s, y) = q(s, y)$  since if  $\dim \tilde{Q}(s, y) = q(s, y) + 1$  then in fact the set in (87) contains  $\text{int } Q(s, y)$ .

By our assumption  $\tilde{Q}(s, y)$  contains at most  $q(s, y) + 1$  linearly independent points in  $\mathbb{R}^{n+1}$ . By convexity,  $\tilde{Q}(s, y)$  must be an affine graph over  $Q(s, y)$ .

Let  $v_0, v_1 \in Q(s, y) \cap \partial A_s$  (necessary in  $\partial A_s \setminus \partial P$  by Lemma 6.1) and assume  $v_0 \neq v_1$ . This is possible since  $\{y_i\}_{i \in I} \subset Q(s, y) \cap \partial A_s$  and  $|I| > 1$ . Consider the function

$$F(a) := u_s((1-a)v_0 + av_1) \quad a \in [0, 1].$$

Denote  $v(a) := (1-a)v_0 + av_1 \in Q(s, y)$  (the inclusion follows from convexity of  $Q(s, y)$ ). Then

$$F''(a) = \langle \nabla^2 u_0|_{v(a)}, (v_1 - v_0), v_1 - v_0 \rangle + s \langle \nabla^2 \dot{u}_0|_{v(a)}, (v_1 - v_0), v_1 - v_0 \rangle. \quad (89)$$

If the statement of the Lemma were false then in fact by continuity it would necessarily follow that

$$\tilde{Q}(s, y) = \text{graph of } -\dot{u}_0 \text{ over } Q(s, y).$$

However, since  $\tilde{Q}(s, y)$  is affine over  $Q(s, y)$  then the second term in (89) would vanish, implying strict convexity of  $F$  on  $[0, 1]$ . However, that contradicts the fact that  $\nabla u_s(v_0) = \nabla u_s(v_1)$  (see (37) and (76)), hence  $F'(0) = F'(1)$ .

Finally, (88) follows from (87) since, if  $\dim \tilde{Q}(s, y) = q(s, y)$ , the fibers of the projection  $\pi_P : \tilde{Q}(s, y) \rightarrow Q(s, y)$  are singletons.  $\square$

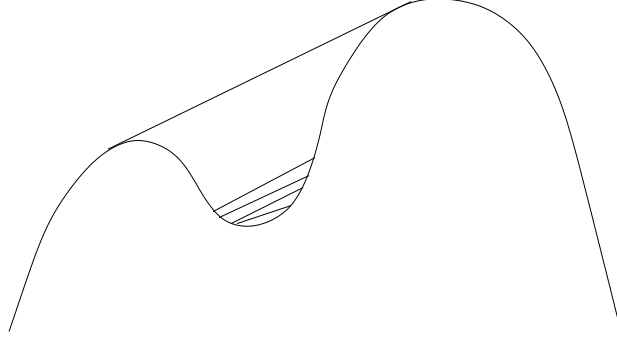


FIGURE 4. The graphs of  $-\dot{u}_0$  and  $-(\dot{u}_0^{**})$  over  $P$ , and line segments in the image of  $\partial\psi$  corresponding to  $\tilde{Q}(s, y)$  for different values of  $s$ .

**LEMMA 13.3.** *Let  $T > T_{\text{span}}^{\text{cvx}}$ . Then*

$$0 < \int_{\Sigma_{\text{sing}}(T)} \text{MA } \psi < \text{Vol}(\text{epi}(-\dot{u}_0) \setminus \text{epi}(-(\dot{u}_0)^{**})). \quad (90)$$

*Proof.* Let  $s > T_{\text{span}}^{\text{cvx}}$  and let  $y \in A_s \setminus \partial P$ . Note that we have a partition

$$\overline{A_s} \setminus \partial P = \bigcup_{v \in A_s \setminus \partial P} Q(s, v), \quad (91)$$

that is any two sets appearing in the union are either disjoint or else coincide. Let  $x := \nabla u_s^{**}(Q(s, y))$ . By Lemma 12.4  $Q(s, y) = \partial_x \psi(s, x)$ , and further by Lemma 12.5,

$$\tilde{Q}(s, y) = \text{co} \{(-\dot{u}_0(v), v) : v \in Y(s, y)\} \subset \partial\psi(s, x). \quad (92)$$

Therefore, from (91) we conclude that

$$\partial\psi(\{s\} \times \mathbb{R}^n) \supset \bigcup_{v \in A_s \setminus \partial P} \tilde{Q}(s, v). \quad (93)$$

According to Proposition 11.1 (i) the set  $\partial\psi([0, T_{\text{span}}^{\text{cvx}}) \times \mathbb{R}^n)$  is equal to the graph of  $-\dot{u}_0$  over  $P \setminus \partial P$ . On the other hand, whenever  $T > T_{\text{span}}^{\text{cvx}}$ , the set on the right hand side of (93) projects to the non-empty set  $\overline{A_s} \setminus \partial P$ . Above each piece  $Q(s, v)$  of the partition of  $A_s \setminus \partial P$ , the set  $\tilde{Q}(s, v)$  satisfies  $\pi_P(\tilde{Q}(s, v)) = Q(s, v)$  and is either

a convex body of one dimension higher than  $Q(s, v)$  or else is an affine graph over  $Q(s, v)$ . We now describe these two possibilities in more detail.

Let  $s \in (T_{\text{span}}^{\text{cvx}}, T)$  and let  $y \in A_s \setminus \partial P$ . Either,

(a)  $\dim \tilde{Q}(s, y) = q(s, y)$ , in which case by Lemma 13.2 there exists a subset  $U(s, y)$  of  $Q(s, y)$ , with non-empty interior relative to  $Q(s, y)$ , such that

$$\tilde{U}(s, y) := \tilde{Q}(s, y) \cap \pi_P^{-1}(U(s, y))$$

is affine and does not intersect the graph of  $-\dot{u}_0$  over  $U(s, y)$ . Or,

(b)  $\dim \tilde{Q}(s, y) = q(s, y) + 1$  (while the graph of  $-\dot{u}_0$  over  $Q(s, y)$  has dimension  $q(s, y)$ ). In this case there exist  $q(s, y) + 1$  distinct points  $\{v_i\}_{i=1}^{q(s, y)+1} \subset Y(s, y)$  such that  $Q'(s, y) := \text{co} \{v_i\}_{i=1}^{q(s, y)+1} \subset Q(s, y)$  is a polyhedron of dimension  $q(s, y)$ , and such that  $\{(-\dot{u}_0(v_i), v_i)\}_{i=1}^{q(s, y)+1} \subset \tilde{Q}(s, y)$ , are linearly independent in  $\mathbb{R}^{n+1}$ . Then  $\tilde{Q}'(s, y) := \text{co} \{(-\dot{u}_0(v_i), v_i)\}_{i=1}^{q(s, y)+1} \subset \tilde{Q}(s, y)$  is a polyhedron of dimension  $q(s, y)$  that can be considered as an affine graph over  $Q'(s, y)$ . Imitating the proof of Lemma 13.2 for  $Q'(s, y)$  instead of  $Q(s, y)$  then implies that there exists a set  $U(s, y) \subset Q'(s, y)$  with non-empty interior relative to  $Q'(s, y)$ , and hence also relative to  $Q(s, y)$ , such that if we set

$$\tilde{U}(s, y) := \tilde{Q}'(s, y) \cap \pi_P^{-1}(U(s, y))$$

then

$$\tilde{U}(s, y) \cap \{(-\dot{u}_0(v), v) : v \in U(s, y)\} = \emptyset.$$

Note that the construction in (b) in effect reduces case (b) to case (a).

Let

$$V(s) := \bigcup_{v \in A_s} U(s, v) \subset A_s \setminus \partial P.$$

It follows from the above and a Fubini type theorem that for each  $s > T_{\text{span}}^{\text{cvx}}$  the non-empty set  $V(s)$  has positive Lebesgue measure in  $\mathbb{R}^n$  (note that, in principle, as defined,  $V(s)$  might not be open): indeed, each  $U(s, y)$  is open relative to  $Q(s, y)$  and by (91) the union of the sets  $Q(s, v)$  over  $v \in A_s \setminus \partial P$  equals  $\overline{A_s} \setminus \partial P$ , that has non-empty interior in  $\mathbb{R}^n$ . We set

$$\tilde{V}(s) := \bigcup_{v \in A_s} \tilde{U}(s, y) \subset \mathbb{R} \times A_s \setminus \partial P.$$

By construction  $\tilde{V}(s)$  is a locally affine graph over  $V(s)$ .

**DEFINITION 13.4.** *For each  $s > T_{\text{span}}^{\text{cvx}}$ , let  $f(s, \cdot) : A_s \rightarrow \mathbb{R}$  denote the unique locally affine function whose graph over  $V(s)$  equals  $\tilde{V}(s)$  and that is affine over each of the sets  $Q(s, y)$ .*

The graph of  $f(s, \cdot)$  is obtained by extending affinely each affine piece  $\tilde{U}(s, y)$  originally defined over  $U(s, y)$  to all of  $Q(s, y)$ . Note that the graph of  $f(s, \cdot)$  restricted to  $V(s)$  satisfies

$$\emptyset = \{(-\dot{u}_0(v), v) : v \in V(s)\} \cap \{(f(s, v), v) : v \in V(s)\}.$$

Further,  $\partial\psi(\{s\} \times \mathbb{R}^n)$  contains  $\tilde{V}(s)$ , i.e., the graph of  $f(s, \cdot)$  restricted to  $V(s)$ .

Now, by Lemmas 8.1 and 8.2 the sets  $A_s$  vary continuously and are strictly monotonically increasing in  $s$ . Also,  $A_{s'} = \emptyset$  for  $s' < T_{\text{span}}^{\text{cvx}}$ . In addition, each piece  $Q(s, v)$  intersects  $\partial A_s$ . It follows that we may assume that  $f(s, y)$  defined above is continuous in  $s$  for a.e.  $y \in A_s$ . Furthermore, it also follows that there exists a set  $I \subset (T_{\text{span}}^{\text{cvx}}, T]$  of positive measure (in  $\mathbb{R}$ ) and subsets  $V'(s) \subset V(s)$  of positive Lebesgue measure (in  $\mathbb{R}^n$ ), for each  $s \in I$ , such that

$$\{(f(s, v), v) : v \in V'(s)\} \cap \{(f(s', v), v) : v \in V'(s')\} = \emptyset, \quad s \neq s', \quad s, s' \in I.$$

Let  $F(s, y)dy$  be the volume measure induced on the graph of  $f(s, y)$  over  $A_s$ , regarded as a hypersurface in  $\mathbb{R}^{n+1}$ , from the Lebesgue measure on  $\mathbb{R}^{n+1}$ . From the above we have  $\int_{V'(s)} F(s, y)dy > 0$  for each  $s \in I$ . Fubini's theorem now gives that

$$\int_{T_{\text{span}}^{\text{cvx}}}^T ds \int_{V(s)} F(s, y)dy > \int_I ds \int_{V'(s)} F(s, y)dy > 0.$$

Hence,

$$\bigcup_{\substack{s \in (T_{\text{span}}^{\text{cvx}}, T] \\ v \in V(s)}} \{(f(s, v), v) : v \in V(s)\}$$

and therefore also

$$\bigcup_{\substack{s \in (T_{\text{span}}^{\text{cvx}}, T] \\ v \in A_s \setminus \partial P}} \tilde{Q}(s, v)$$

contains a set of positive Lebesgue measure in  $\mathbb{R}^{n+1}$ . By (93) this set is contained in  $\partial\psi((T_{\text{span}}^{\text{cvx}}, T] \times \mathbb{R}^n)$ . According to Theorem 2.4 this implies the lower bound in (90).

The upper bound in (90) follows from Lemma 8.1 and the upper bound of Lemma 12.5.  $\square$

Lemma 13.3, together with Proposition 11.1 conclude the proof of Proposition 13.1, from which Theorem 1 follows.

**Acknowledgments.** This material is based upon work supported in part under National Science Foundation grants DMS-0603850, 0904252. Y.A.R. was also supported by a National Science Foundation Postdoctoral Research Fellowship at Johns Hopkins University during the academic year 2008–2009.

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